Supplement to Beyond the Box: Cardboard Math-Art

George Hart and Elisabeth Heathfield

This supplementary document provides scalable cutting templates for the fourteen cardboard constructions illustrated in our paper for the 2025 Bridges conference:

> G. Hart and E. Heathfield, "Beyond the Box: Cardboard Math-Art",
> *Proceedings of Bridges Eindhoven 2025*, Tom Verhoeff et al. eds.

We also include some assembly tips and references with useful background information for anyone wishing to replicate these constructions or to delve deeper into their mathematical foundations. But some experimenting will be required as each is something of a puzzle to conceptualize and construct.

The templates for each appear on a separate page, following the sequence of the figures in the main paper. Those figures should also be consulted for additional guidance.

For cutting cardboard we mostly use a laser cutter, but sometimes use a band saw or scroll saw. Knives are not recommended because experience proves they lead to accidents. In the templates below, cutting lines are shown in red. A red (laser-cut) dotted line removes enough material to allow a straight fold while leaving enough that the material remains attached. Some templates also include blue lines, which are for marking or folding. For them, the laser power should be set low enough (or the speed high enough) that the cardboard is only scored.

Cardboard comes in various thicknesses and degrees of stiffness. You can adapt these designs to your materials and chosen scale. The templates are provided as vector graphics in this PDF supplement file so they may be extracted and scaled for laser cutting at any size without loss of accuracy.

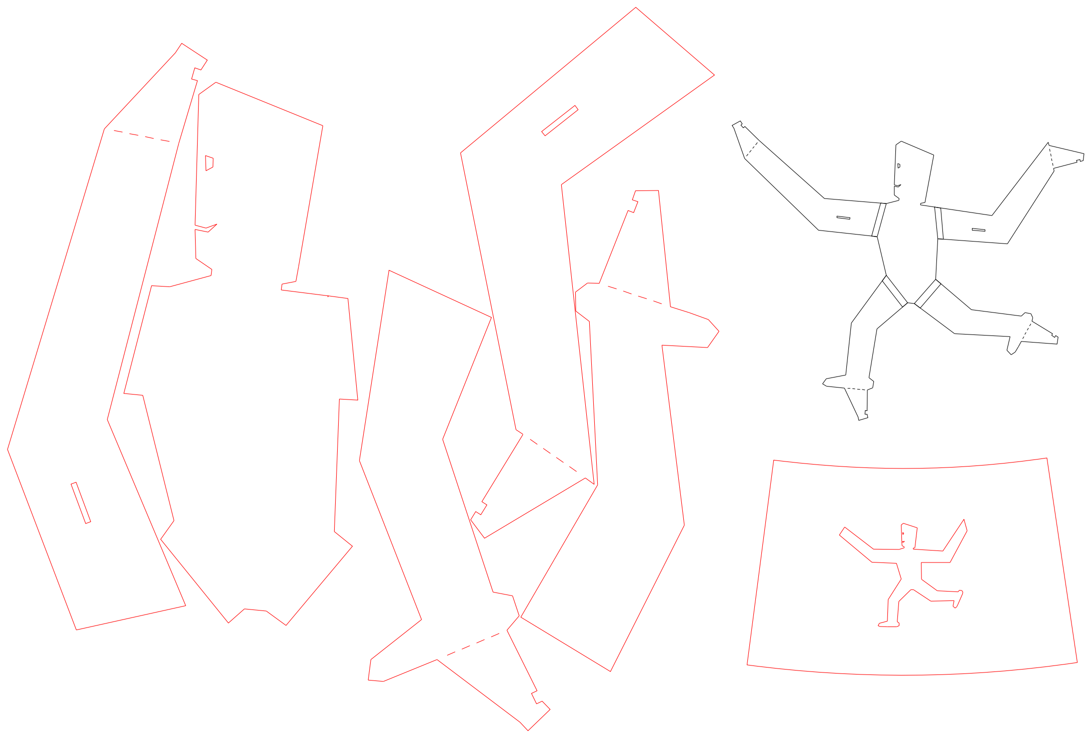
For gluing, we find a thin layer of Titebond II glue dries quickly to make a very strong joint. (Too much just makes a mess and is not any stronger.) Craft stores and carpentry suppliers sell small glue brushes that work very well for spreading glue. We recommend construction workshop participants use a scrap of cardboard as a palette to hold a small supply of glue for brushing. Black binder clips, available in bulk at office supply stores, work well to hold pieces firmly together while the glue is drying. The handles of these clips can be folded "up" or "down" and this provides a convenient marker to show which joints have been glued; simply leave the handles up when first assembling, then fold them down after each joint is glued. Typically 15 to 20 minutes is sufficient for the glue to set, but allow an hour in larger constructions where the joints are more strained. After the glue is set, the clamps are removed and collected for reuse on other constructions.

Note that two of these designs use slots, so require no glue (Figures 10 and 17). For two others we recommend using tape (Figures 8 and 13). And for one we recommend hot-melt glue (for its rapid setting time) instead of liquid glue (Figure 5).

These designs vary considerably in difficulty and it is best to begin with simpler examples to gain experience with the techniques. The approximate ranking from simplest to most difficult is:

10, 2-3, 6-7, 13, 4, 8, 9, 17, 14-15, 16, 12, 5, 11, 1

Figure 1. "People" sculpture



This is the largest and most complex cardboard sculpture we have assembled. We recommend you do not attempt it before gaining experience with several simpler examples.

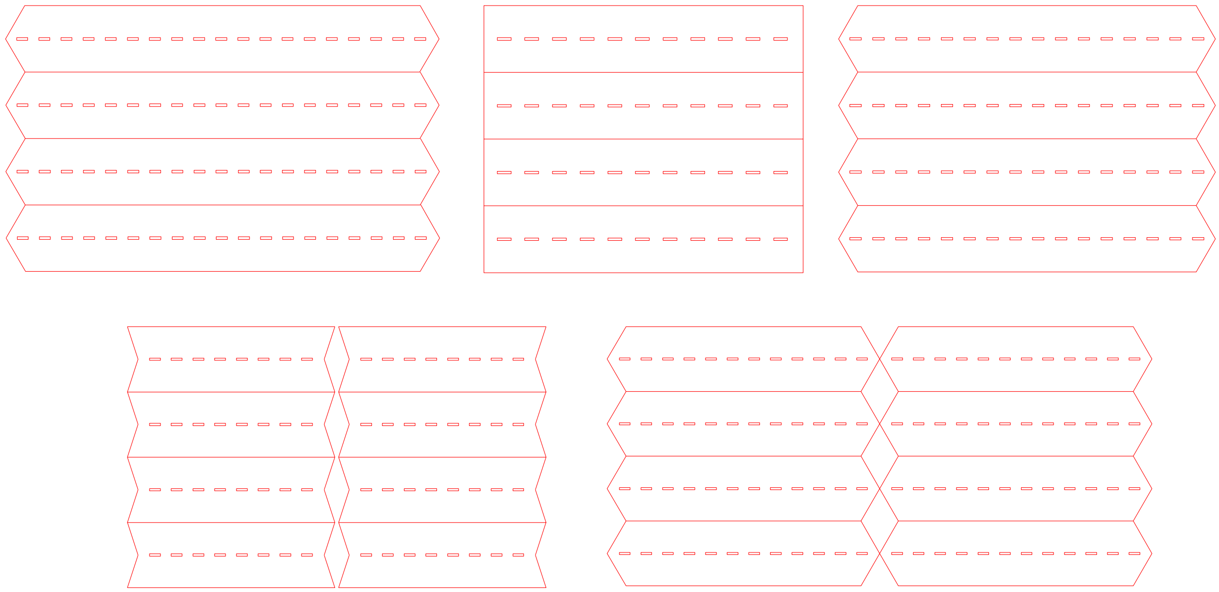
Sixty "people" are required. Because the entire shape did not fit on any laser cutter available to us, each piece is assembled from five components as indicated in the inset diagram. Note the front and rear legs are subtly different in the angle of the ankle. The example shown in the paper was scaled so the height of the torso piece is 17 inches. There are four quadrilateral areas of overlap where glue is applied when attaching the limbs to the torso. The tabs at the ends of the legs and arms are to be bent backwards, joined in pairs, then inserted into a shoulder slot. With "front" and "back" defined by the face direction, a front arm always meets a front leg and their tabs go into a front shoulder slot. A rear arm always joins a rear leg and their tabs go into a rear shoulder slot.

The five body components are shown to the same scale but the "curved rectangle" base piece is shown at a reduced scale. In our construction it was 11 inches tall. Ten are required.

Some mathematical aspects of this design are discussed in:

G. Hart, "Sculptural Presentation of the Icosahedral Rotation Group", *CRM-AMS Proceedings & Lecture Notes series, Groups and Symmetries Conference, 2019,* AMS publications, p. 211-214.

Figures 2 and 3. Platonic solids edge models



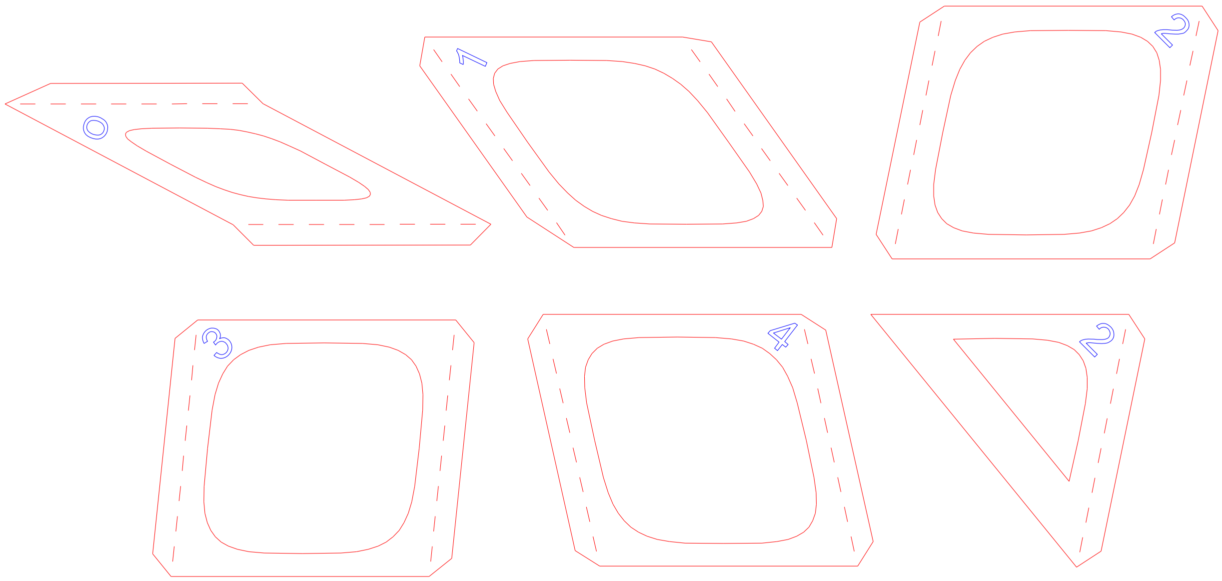
Top row: *tetrahedron (6 needed), cube (12 needed), octahedron (12 needed)*
Botom row: *dodecahedron (30 needed), icosahedron (30 needed)*

This construction makes large, strong, open-face models of the Platonic solids and is an excellent introduction to the clamping-then-gluing technique. For the workshop shown in the paper, these templates were all scaled so the cube edge is 14 inches long. The parts shown above are each grouped into stacks four strips high, because that is just under one foot at the scale we used, which was convenient for our laser cutter and cardboard sources. The caption states how many edges are needed for each Platonic solid.

To help distinguish these unlabeled parts, note how the ends are trimmed to make a 60 degree angle for the three polyhedra with equilateral triangular faces, to 90 degrees for the cube, and to 108 degrees for the pentagonal faces of the dodecahedron. To distinguish the tetrahedron, octahedron, and icosahedron pieces, note that the more edges in the polyhedron, the shorter the individual edges are.

Each strip should first be folded lengthwise and creased firmly to eliminate any springiness that would work against the polyhedron holding together. Each model should be completely assembled with small binder clamps and inspected visually for accuracy before applying glue. The corners should be snugged together tightly. They look neatest if the overlaps of the edges cycle consistently, e.g., A over B over C over A. With participants working in pairs, the clamps can be removed from one vertex by one partner, allowing the other to brush in glue; then the first replaces the clamps before moving on to the next vertex. The completed vertices can be marked by the position of the clamp handles, making it easy to find those which still need glue.

Figure 4. Zonohedron Dome



For an introduction to the structure of polar zonohedra, see the reference below, which also describes how these rhombus angles are calculated.

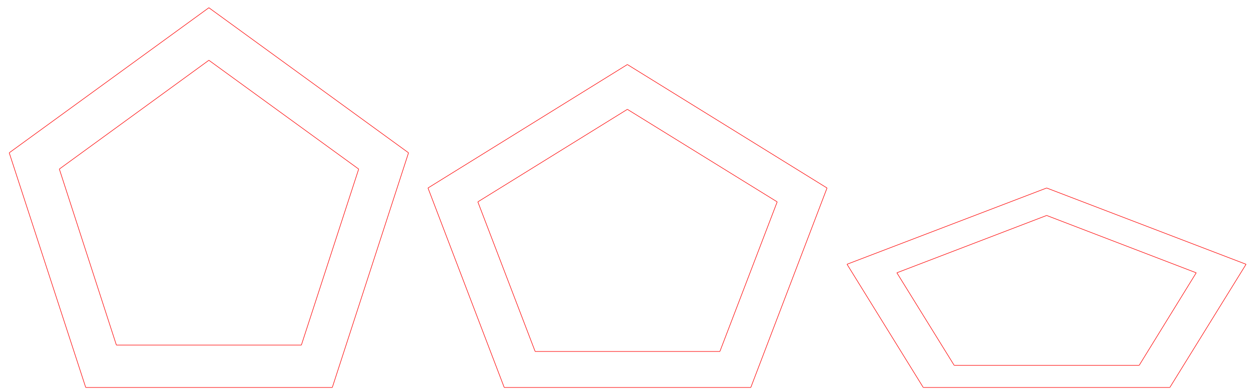
This is a 10-fold polar zonohedron, so there are ten rhombi in each layer. The peak consists of ten rhombi of shape 0 which all meet at the upper pole. Below that are ten of shape 1, then ten each of shapes 2, 3, and 4 at the widest level. Below that the form tapers in with another layer of shape 3 and to finish the bottom smoothly, a layer of the half-rhombi of shape 2. So ten pieces of each of the above shapes are required, except that twenty are required for shape 3.

Each piece should be oriented so that the vertex with the number label is uppermost. The connecting flaps can be folded back slightly along the dotted line then clamped to the inside of the adjacent face. The flaps were made on only two of the four edges of each rhombus and they alternate right-left-right-left on the layers so that two flaps never interfere.

For the example in the paper, the template above was scaled so that the edges of each rhombus are 12 inches (ignoring the flaps). In the orientations above, the flaps do not extend the height of any piece beyond 12 inches, so they can be cut on a laser cutter that has a 12 inch maximum height.

G. Hart,"The Joy of Polar Zonohedra," in *Proceedings of Bridges 2021,* David Swart et al. eds,, pp. 7-14.

Figure 5. The 120-Cell (Cell-First Projection to 3D)



Call these three progressively shortened hollow pentagon shapes A, B, and C. Laser-cut ahead of time 36 of type A, 90 of type B, and 150 of C. For the construction shown in the paper, this template was scaled so that the regular pentagon, A, has edge length 4.75 inches.

An excellent way to learn about the beautiful 4D structure of the 120 cell is to make a cell-first 3D projection using the ZomeTool plastic construction system. Instructions are given in the references below. In four dimensions all the cells are regular dodecahedra, but in this three-dimensional shadow, the cells have been overlapped and compressed into five different dodecahedral shapes. The central dodecahedron is regular and as you move out to the periphery the others are progressively foreshortened in a way that is best understood with a ZomeTool model.

The pieces we are working with are hollow laser-cut cardboard pentagons. Shape A is a regular pentagon. B and C are progressively more foreshortened, with the same base edge length as A. The assembly technique is to use a glue gun to apply a small dab of hot-melt glue in each corner of a pentagon. One person can hold a pentagon in place while another applies the glue. The piece must be held still for about a minute while the glue solidifies. The result is very strong, extremely rigid, and lightweight. Although the structure is quite complex, a group construction is faster than one might think because several participants can work on different sides at once. Throughout the interior, three pentagons butt together at each edge and the correct choice of piece type and position takes some geometric intuition.

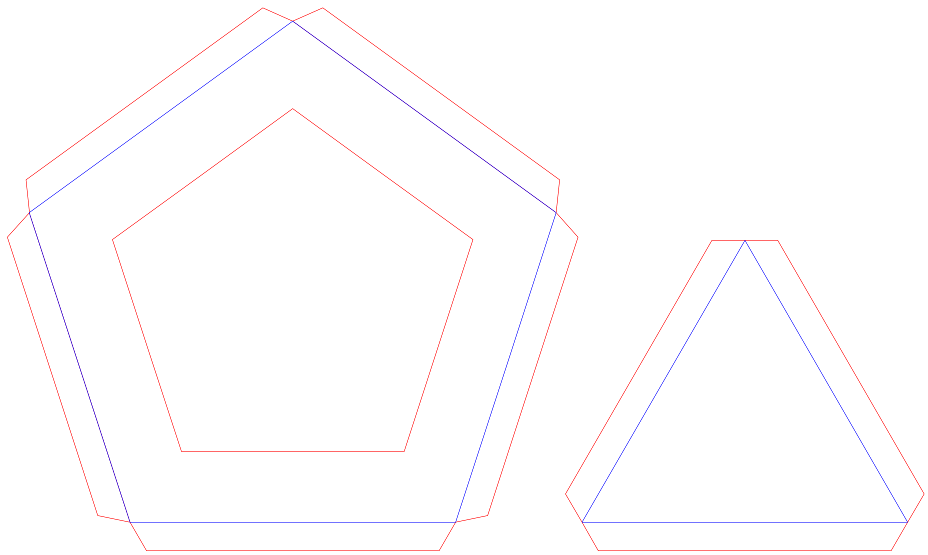
The construction starts by making the central regular dodecahedron using twelve faces of type A. Pentagons are then added individually to the ever-growing structure. (So don't try to make two halves and join them!) Each of the second layer of cells consists of two A faces and ten B faces. Add a part B to each edge of the central cell. This makes a kind of five-sided bowl which is then easily completed into a cell with six more pentagons (five more B and one A). The third type of cell consists of six B's and six C's. The fourth cell type has ten C's and two A's. The outermost cells are completely flattened, but do not need to be specifically constructed, because they form themselves when the other other cells are built.

Other polytope models in the H_4 family can be built with the same technique. Be the first to try them!

G. Hart and Henri Picciotto, *Zome Geometry: Hands-on Learning with Zome Models*, Key Curriculum Press, 2001.

G. Hart, "Barn Raisings of Four-Dimensional Polytope Projections", in *Proceedings of International Society of Art, Math, and Architecture 2007*, Texas A&M, May, 2007.

Figures 6 and 7. Icosidodecahedron



The templates here are just an open-face regular pentagon and a solid equilateral triangle, each extended slightly with folded flaps for connecting to the adjacent faces. Twelve pentagons and twenty triangles are needed for the icosidodecahedron shown in the paper.

The challenging part of this construction is to scale these up to a giant size. One option is to use traditional straightedge and protractor methods to draw the templates on the cardboard. Another option is to print an enlarged template on to multiple sheets of paper that are cut out and taped together (as seen at left in Figure 7 of the paper), which is then traced onto the cardboard.

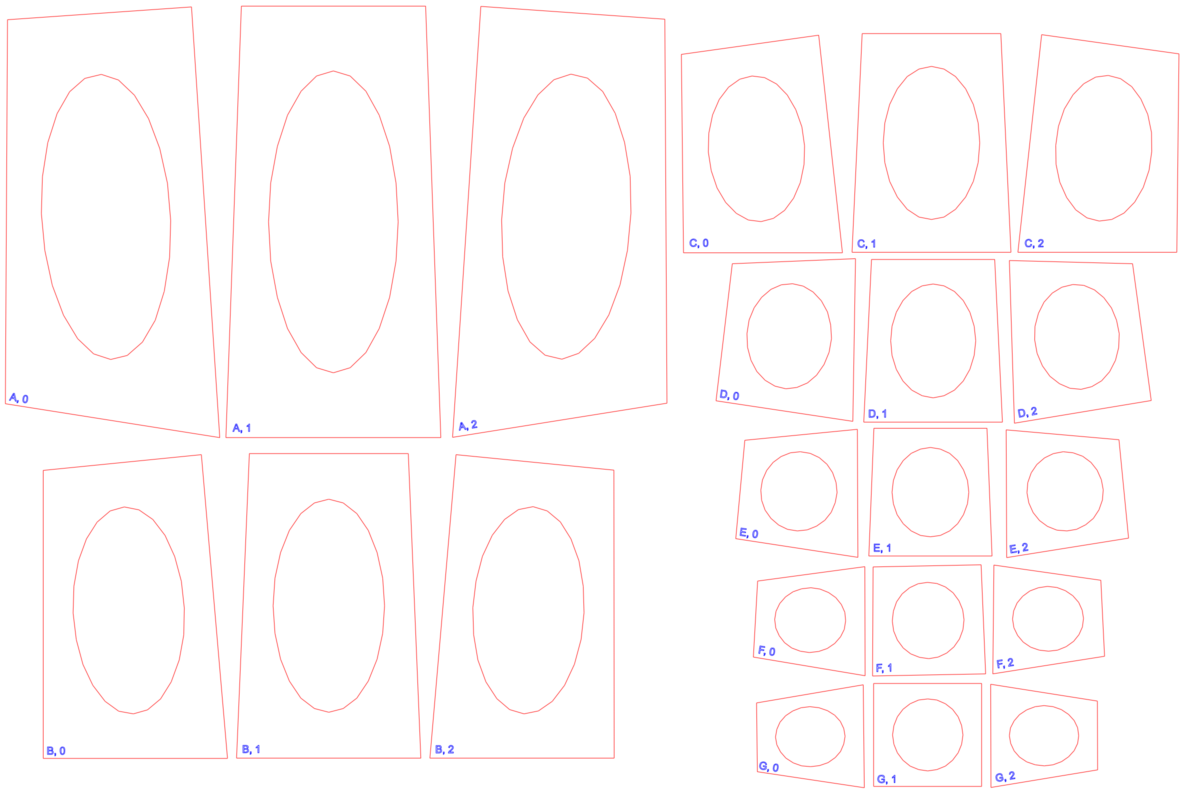
We chose a 16-inch edge length, making the pentagon piece over 26 inches high (including the flaps). This was too large for us to laser-cut, so we used traditional wood-shop equipment. First we stacked six to eight layers of cardboard into a thick pile and held them together with a few sheet-rock screws so we could cut a whole stack at a time. (Choose the screw length and number of layers to match, i.e., the screws should just reach to secure the bottom sheet without protruding through the bottom of the stack.) Trace the template on the top sheet of each stack. The outside cuts of these stacks are easily made with a band saw. For the interior cuts of the hollow pentagon, use a hand jig-saw.

To fold the flaps, use the edge of a table as a guide by positioning each piece so a fold line is above the table edge (with just the flap protruding) and pushing the flap downward.

To assemble the pieces, position them with their flaps on the inside, paired along the inside of each edge, and fasten each pair of flaps with binder clips.

With some care, these pieces can be reused to make a variety of structures including other Archimedean solids.

Figure 8. Catenary Arch



We scaled these pieces so the largest (A1) just fit in the 12×24 inch bed of our laser cutter. Two of each of the A through F pieces are needed, but only one of each of the three G pieces is needed.

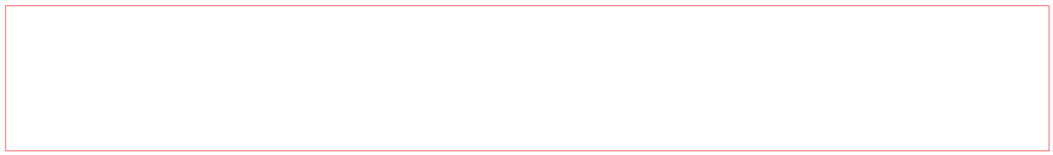
The design ideas are explained in the reference below, but here we are using cardboard at a large scale and no triangular support pieces are needed.

The assembly method is to use clear packing tape along all the edges. Three pieces A0, A1, A2 are joined into an approximate triangular prism; three pieces B0, B1, B2 are joined into a smaller one, etc. Then the prisms are taped together, from A up to F for the two legs, with the G prism connecting the legs at the top. Be sure the labeled corner is always at the lower left and the sequence 0, 1, 2 is always maintained. The pieces numbered 1 will form a band along the outside of the arch.

For stability after it is all assembled, the A units should be taped to the floor.

G. Hart and Elisabeth Heathfield, "Catenary Arch Constructions," in *Proceedings of Bridges 2018*, Eve Torrence et al. eds., pp. 325-332.

Figure 9. Regular Polylink of Six Squares



The template here is simply a rectangle with sides in the ratio of 1 to 7.2, scaled up to any convenient size, perhaps two to three feet long. Twenty four of these rectangles are required. Although the finished structure seems simple enough, consisting merely of six squares arranged symmetrically in the planes of a cube, the assembly is a surprisingly tricky visualization and dexterity exercise. Detailed instructions for a smaller craft-stick-size version of the structure are given at the references below. They can be followed directly for this larger size. Working in a team is very helpful for manipulating these larger pieces.

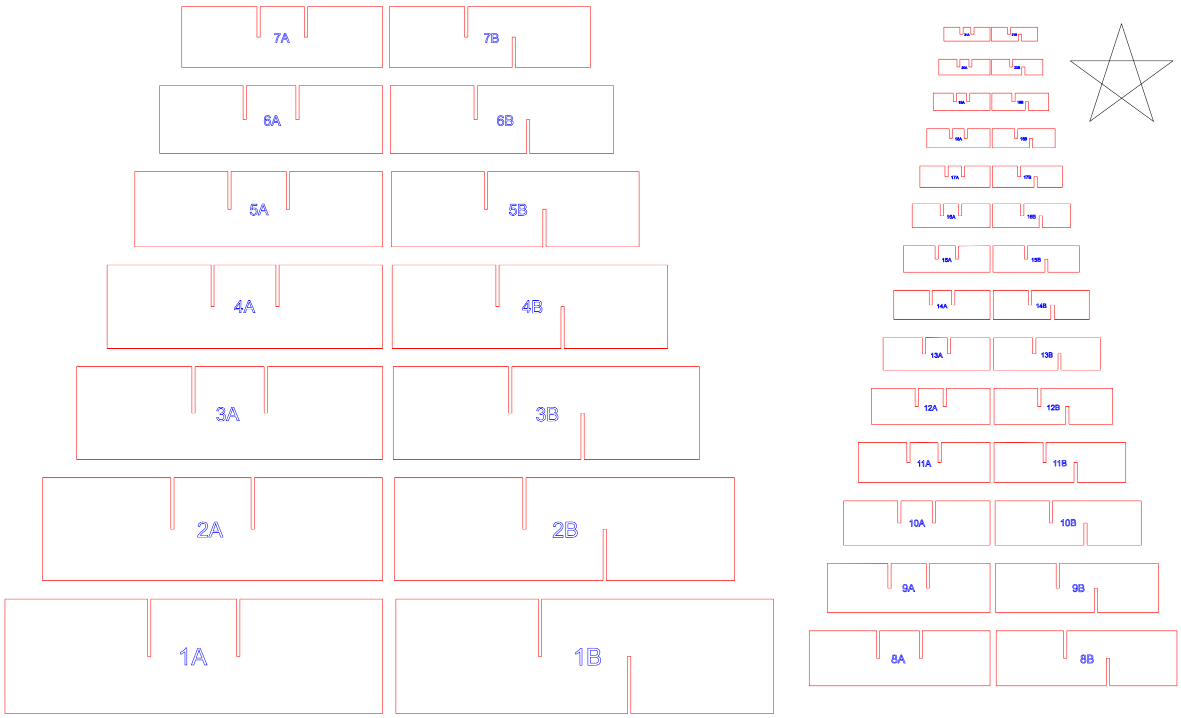
The result looks best if the four overlaps of each square are symmetrical and identical, i.e., A over B over C over D over A in cyclic order. Check for this after clamping. Rearrange if necessary before gluing.

G. Hart, "Orderly Tangles Revisited", Proceedings of *Bridges 2005,* Banff, AB, 2005.

https://makingmathvisible.com/Polylinks/polylinks.html

https://www.youtube.com/watch?v=ceY94c8OXnU

Figure 10. Tree

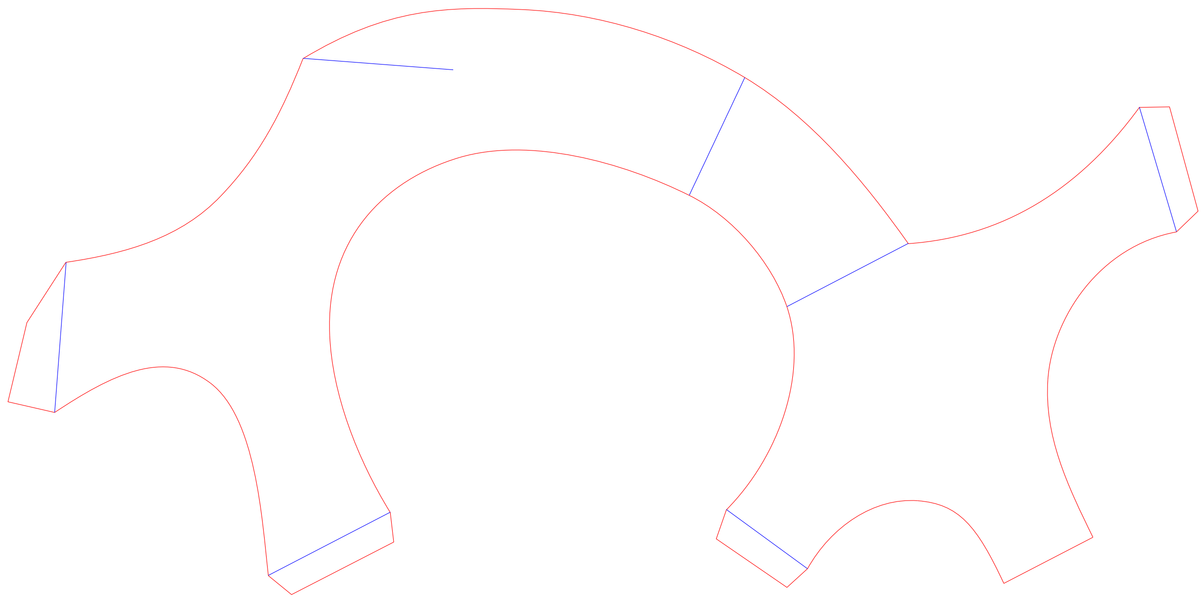


We scaled these templates so that pieces 1A and 1B are each 24 inches wide.

Each level of this 21-layer "tree" consists of five slotted rectangular pieces, two of type A and three of type B. The top view of each layer is a pentagram star, as shown in the inset above. The layers are built independently then stacked from number 1 on the bottom to number 21 at the top. It is a small puzzle (with several possible solutions) to arrange the five pieces in each star, but one quickly learns to save an A piece for the end, because its two slots face the same direction, allowing it to be slid in last.

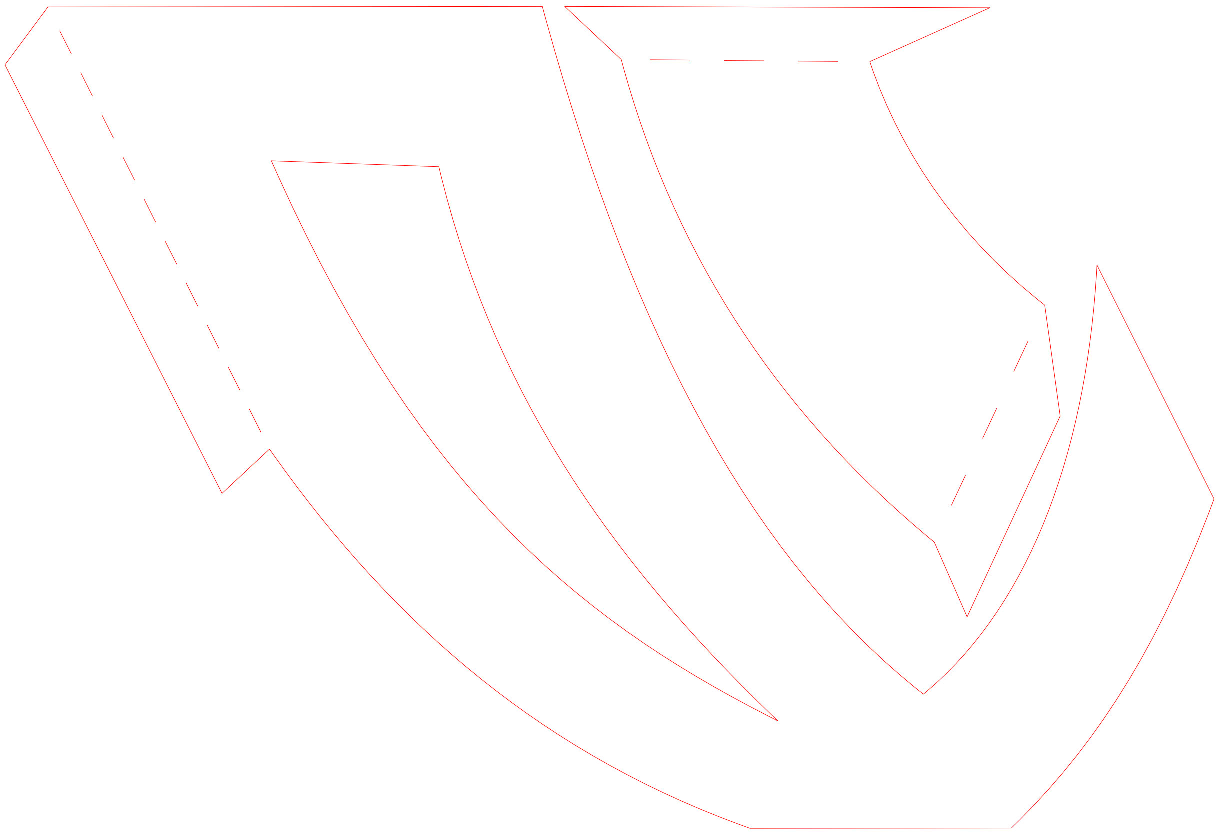
No glue or tape is required. The layers can simply be stacked, with the points of each alternate layer pointing in opposite directions.

Figure 11. Untitled Sculpture



This sculpture is rather complex to describe, so we leave its assembly as a puzzle for the adventurous. Sixty identical components are required to make one orb. The three interior blue segments are not fold lines; they are alignment marks that indicate where flaps from other pieces connect.

Figure 12. Autumn

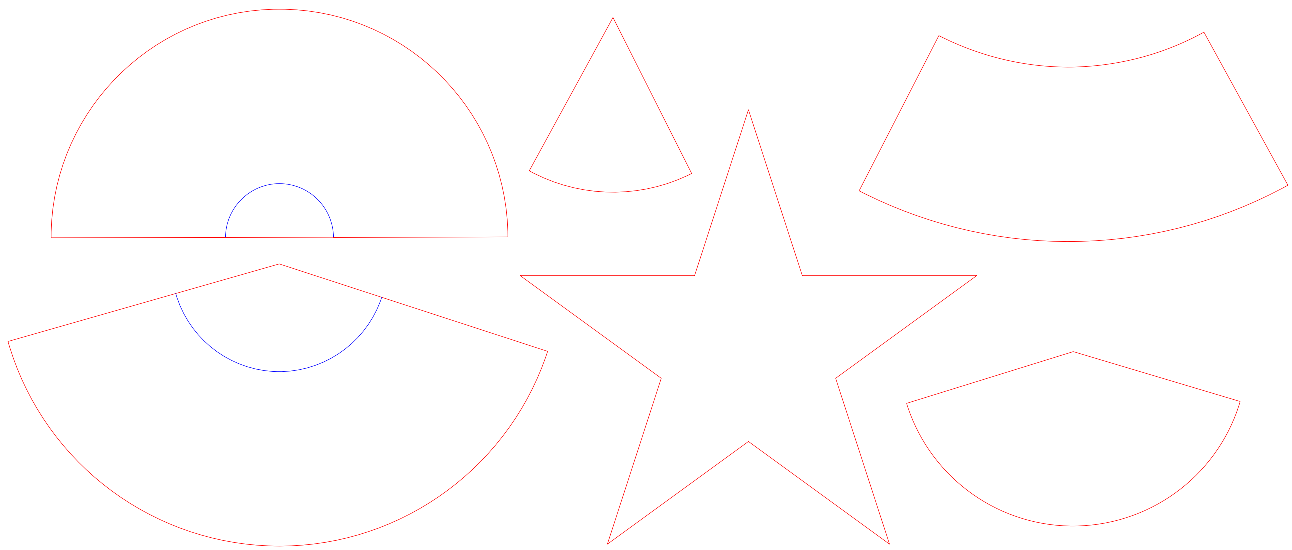


Sixty of each piece is required to make one orb. For the two orbs shown in the paper, the template above was scaled so the long edge of the larger piece (the horizontal line at top left in the orientation above) is 7 inches.

The assembly follows the steps shown for the wooden *Autumn* sculpture on the Making Math Visible link below, with one addition. The larger piece here corresponds to the wooden piece there and is assembled similarly in groups of three by gluing the long flap of each to the interior of its neighbor. The smaller piece here is added as an internal brace and connector, needed for rigidity because of the much larger scale. So twenty modules are first made, each consisting of three of the larger pieces and three braces. There is only one way they can connect with the brace's fold lines meeting straight edges of the larger pieces. Use the clamp-then-glue method, first checking that all the flaps are interior before brushing in the glue. When the twenty modules are dry, they can be joined together by clamping the brace of one module to the brace of its neighbor, making cycles of five modules.

https://makingmathvisible.com/Geometric-Sculpture/sculptures.html

Figure 13. Streptohedron Based on Pentagram



These six pieces can be scaled to any convenient size. To make the half-streptohedron shown in the paper, one of each piece is required. The conical surfaces can be gently formed with a series of slight folds until they have the desired curve. The star and semicircle remain planar. Start by attaching the semicircle and largest wedge to the star. Everything should be taped together securely on all edges. The blue arcs are alignment marks that indicate where the next two pieces will connect. The piece shown at bottom right above becomes a concave surface – half a cone – opposite the apex. If two halves are made, their star faces can be taped together temporarily (e.g., with painter's tape) in five different orientations to see the various ways that the combined object rolls down a slightly inclined plane.

For background, see this reference, especially Figure 10 and Photo 5:

David Springett, "Streptohedrons (Twisted Polygons)", *Proceedings of Bridges London 2006*.

To calculate the various lengths and cone angles, e.g., as a classroom activity, one must first work out that the diagonals of a pentagram cross each other at the golden ratio points. Then multiple applications are used of the formula for the curved circumference of a circular arc of given central angle ($c = 2\pi r\,\theta/360$).

Figure 14-15. Orb with Non-Edge-to-Edge Tiling of the Sphere

The pieces for this giant orb are 4×48 inch rectangles. 120 were used because they were laminated in groups of three to make 30 large triple-thickness arcs. A simple jig was used while the glue dried to hold the pieces in the shape of approximately a quarter circle of roughly 2.5 foot radius. (We used a jig consisting of four right-angle pieces glued to a scrap of cardboard as seen in Figure 15 of the paper, but four stacks of heavy books might work just as well.) The one-third and two-thirds point of each arc was marked as a guide for clamping (and later gluing) everything together.

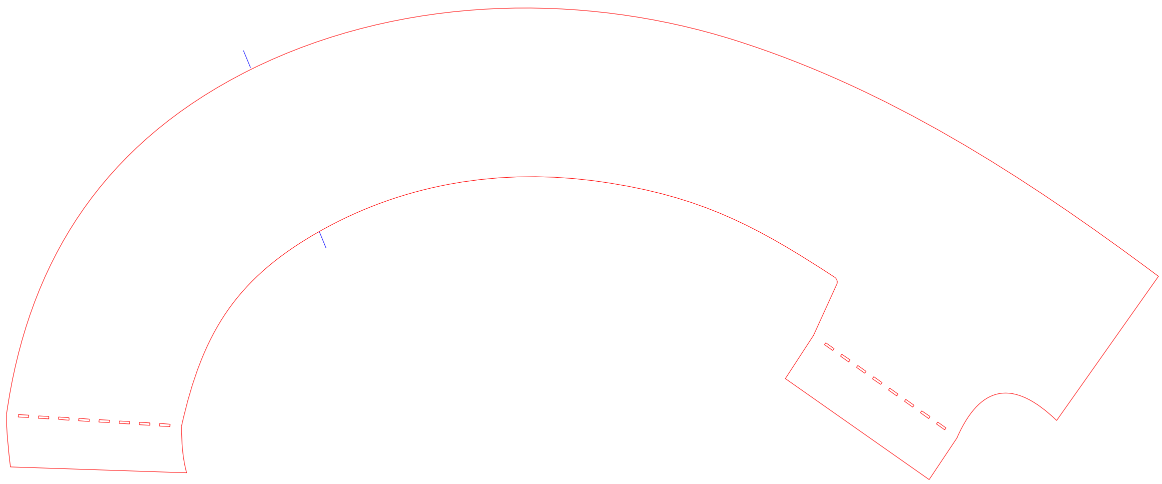
The best way to understand the structure might be to first make a small paper model. Detailed instructions are given here:

G. Hart, "Loopy", *Humanistic Mathematics*, June, 2002, issue # 26, pp. 3-5.

The family of structures has been discovered independently several times and is described under several names. In the architecture literature this structure is sometimes called a "rotegrity" or "nexorade". For a mathematical analysis, see:

Colin C. Adams, Cameron Edgar, Peter Hollander, and Liza Jacoby, "The Non-Edge-to-Edge Tilings of the Sphere by Regular Polygons", Discrete & Computational Geometry 72(3):1029-1085, September 2024.

Figure 16. Sculpture ("Spring")



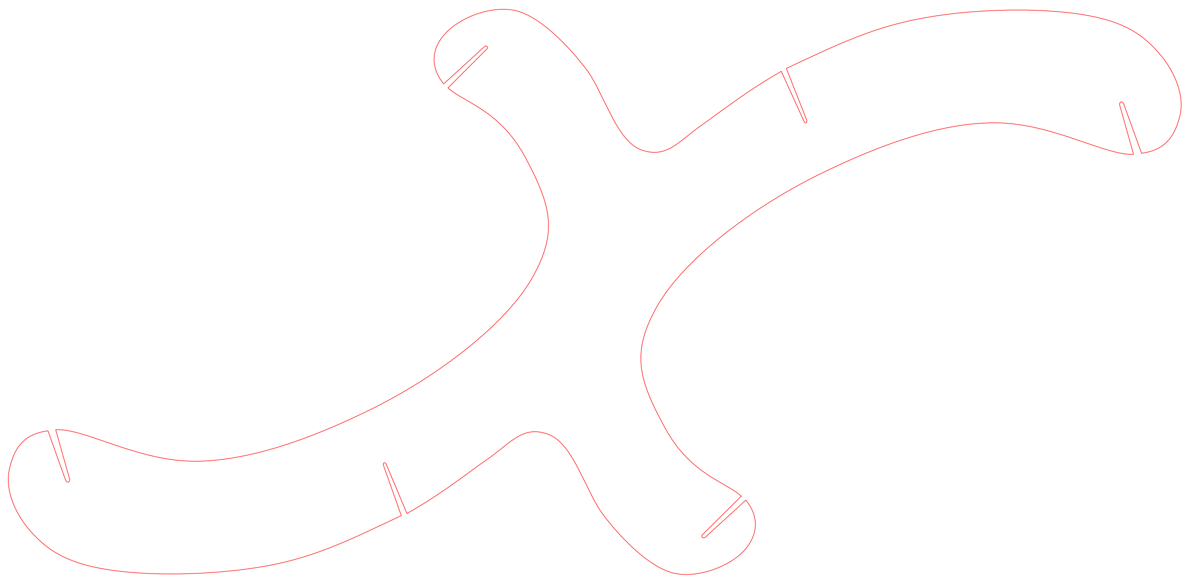
Sixty of these pieces make a large cardboard version of the wood sculpture called *Spring* that is detailed on the Making Math Visible site. (See the link below, *Spring*, Extension A.") If this template is scaled to be 22 inches long, the final sculpture will be about 48 inches in diameter. Note the two blue tick marks above; they indicate where an alignment mark should be made on each piece that show where to make the connections.

https://makingmathvisible.com/Geometric-Sculpture/sculptures.html

The assembly of a cardboard version is shown here:

https://georgehart.com/sculpture/Maker-Faire/cardboard.html

Figure 17. Sculpture



This template can be scaled up to be roughly three feet long. Thirty pieces fit together by joining pairs of slots. Because of their large size, these pieces were cut on a band-saw with the screwed-together stack technique described above . (See Figures 6 and 7.)

Note there are three types of slots: (*a*) ones at the ends of the long arms, (*b*) ones at the ends of the short arms, and (*c*) ones midway along the long arms. Each piece connects with six different neighboring pieces following the rule that slot type *a* always joins with another slot *a*, and type *b* always joins with type *c*.

One can cut a set from paper, scaling this template to 5 or 6 inch size, to practice the construction before building a large cardboard version. A slight flex of the material is required to connect the slots of the final piece. A paper version is shown (in yellow) here:

https://gallery.bridgesmathart.org/exhibitions/2013-bridges-conference/george-hart