Workshop on Impossible Paper Folding: Curves Over Pleats

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Abstract

In this workshop, I show how to fold freely drawn curves over a field of pleats, a technique which enables relatively general "drawing" by folding, while keeping the resulting sheet mostly flat. I developed this technique in recent years and have used it to create various designs and patterns which can be used on their own as reliefs, or as surface decorations for freestanding sculpture. Instead of pleating, crumpling can be used as an alternative in forming the ground over which curves are folded.

Introduction

This paper describes an approach to folding origami models that appear to incorporate curved creases in a way that violates the standard assumption that folding a paper sheet doesn't change its developability. In particular, the models seem to include features such as curved ridges coming out of flat areas. While there are several plausible explanations for how this is possible, it is not completely clear what is actually going on. I have come across this approach to folding through various experiments with origami, most of which involved tessellations based on simple repetitive pleating and the exploration of three-dimensional forms such models take on, either on their own, due to forces generated by pleats held flat by other folds, or due to the flexibility the pleated sheet is afforded by the paper hidden within the pleats.

The models and sculptures whose patterns and construction process I describe here have been developed over the past five years or so. Even though there is a "first," a specific model from 2022 (shown top left in Figure 1) that helped me understand the generality of this approach, I have found examples with similar properties in my work going back a few years further than that. Some examples from 2018 I folded from woven metal mesh, and because mesh can be deformed through shear in ways that would tear paper, at the time I believed they would not be foldable from paper, so I did not try.

I have shown many of these models at origami events and in art exhibitions, and have taught the basics of the approach on several occasions, most recently at 8OSME in Melbourne in 2024. In the 8OSME paper [5], I asked the question (still unanswered), "What is actually going on?" In the current paper, I describe in more details the process of folding such models, and the various possibilities that may be of interest in developing new variations or using existing ones as design elements for more complex pieces.

Traditional origami and most models created until the late 20th century use straight-line folds and result in forms with straight edges and flat surfaces. Before the work of several mathematicians and computer scientists, such as Ron Resch and David Huffman, the only exceptions were models from the Bauhaus school, based on concentric circles, and origami figures by Akira Yoshizawa shaped using wet-folding. Contemporary books by Ogawa [12] and Londenberg [8] also include some curved folds among various other examples of paper sculpture. Surveys of this early work can be found in the paper by Demaine et al [1], and the Ph.D. thesis by Koschitz [6]. If materials other than paper are considered, the history is longer: images and instructions for folding napkins or tablecloth into complex forms and whole elaborate scenes go back at least as far as the 17th century [13]. More recent work on curved crease folding includes a large number of scientific papers, which I do not attempt to survey here, and several collections of models with instructions and folding patterns [3, 11, 10, 9].

Because paper doesn't stretch by any appreciable amount, the standard mathematical model assumes that the folding preserves the intrinsic metric, in other words that Gaussian curvature is zero everywhere and the folded sheet remains developable. One implication of this is that the regions of the sheet adjacent to a curved crease cannot be flat, which the models described and shown here seem to contradict (curved ridge adjacent to the flat base).

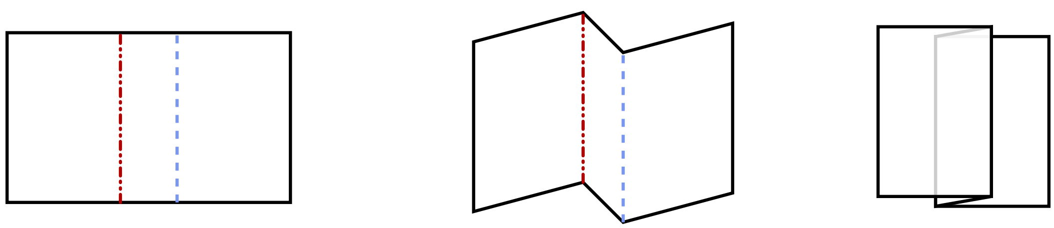
Of course, nondevelopable surfaces can be approximated, and in fact the program Origamizer [2] can be used to create a pattern of folds that, when folded, will collapse into a given piecewise linear surface. This program takes as input a mesh representing the surface and computes the pattern, which due to the precision requirements is usually scored by machine, and even then is difficult to fold. While the basic approach I describe here is more limited with respect to the designs and forms it allows, it is also completely intuitive. Standard methods for origami design include the tree method and box pleating with various generalizations [7]. In both of these design approaches, the process starts with sketching out major desired features in a diagrammatic tree-structured representation. In the tree method, this representation is then used as the input to a nonlinear optimizer. Some constraints on the placement of vertices and various forms of symmetry can be specified as part of the input. The box pleating approach breaks down naturally into several steps, enabling a more interactive design process. The tradeoff is a reduction in efficiency (in terms of finished model size as compared to original unfolded sheet size). The approach in this paper can be seen as an extreme case of this tradeoff, where a simple, mostly fixed, pattern is folded first. This transforms a flat sheet of paper into one that, while still flat (in theory, though not always in practice), is much more flexible and allows new ways (to paper, at least) of sculpting.

Examples

Figure 1: *Some examples of folding curves over pleats.*

Figure 1 shows several early examples, all from a series of small pieces I created in the summer of 2022. (For reference, the finished size of each of these is a 15×15 cm square.) These are some of the first curve drawing designs from the time I began understanding the generality of the approach. The top left example (blue) is the one that motivated the whole approach. It, as well as its neighbor, uses a ground folded from alternately tilting pleats, and very gentle curves that meet the pleats at close to right angles. The third example in the bottom row was folded from a rectangle pleated along lines radiating from a point just outside the sheet; the curves all hit the pleats at right angles, so this is geometrically the simplest case. The red spiral uses the same approach, but with a much longer rectangular strip whose pleated ends overlap and are glued together before forming the curves. The two pieces on the lower left show the first experiments with closed curves. In both cases, the ground consists of pleats laid out in a square grid, but for the two intersecting circles, I rotated the grid to better match the extent of the circles. A completely different way of providing the ground for a closed curve is shown in the third photo on top, with an ellipse inside a circle. If you look carefully, the pleats radiate from a single point in the interior of the sheet, which means that there must have been an excess of paper in the middle before the folding started. I constructed such a sheet of paper by attaching several pieces together to form a non-Euclidean sheet, shaped like a hexagon with right angles at all six vertices, so that the total angle around the center was 540 degrees. (In principle, the effect of a radial pleat center in the interior could also be achieved using a strip, similarly to the red spiral, but that would be even more difficult to fold accurately enough.) Finally, the top right shows four sets of curves folded over a square grid ground.

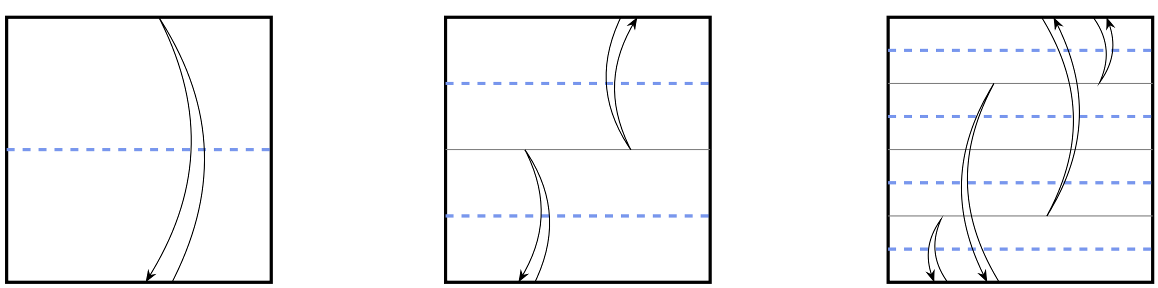
Pleated Ground



Pleat crease pattern | Partially folded pleat | Flat-folded pleat

Figure 2: *A basic pleat consists of two parallel folds of opposite types.*

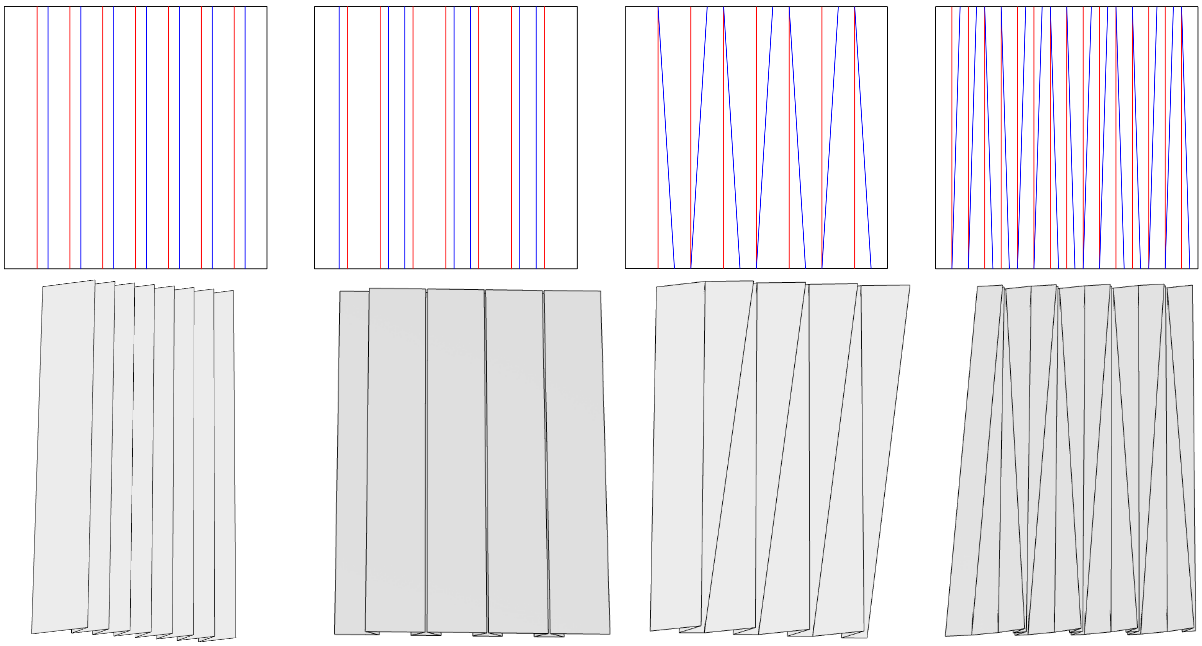
The first step is to pleat or otherwise transform the starting flat sheet of paper into one that can be formed and stretched as needed. In most cases I use a regularly spaced set of parallel pleats, but the uniformity is not important. Pleats that vary in width or density work equally well, as do those whose mountain and valley folds are not parallel, or even collections of pleats that are not parallel to one another. In the extreme case, even crumpling can be used. The one property common to all of these forms of pre-folding the sheet is that there is enough paper hidden in the folds.



To crease into halves: Align the edges, crease and unfold. | To crease into fourths: fold to the center and unfold. | To crease eighths: fold to existing creases and unfold.

Figure 3: *Dividing into equal parts by repeated subdivision. All creases are valley folds. Accuracy may be reduced when the number of divisions exceeds 32 or 64, but that doesn't matter here.*

To fold a pleated ground, I first divide the sheet into (usually) equal strips by dividing it into halves, then fourths, etc. as shown in Figure 3. I turn the paper over, so the creases are all mountain folds, and form the pleats one by one. For each pleat, I use an existing mountain crease as the mountain fold of the pleat. To form the pleat's valley fold, I pinch together the two layers that form the mountain fold, and move them towards the next mountain crease. Typically, I fold the pleat so that its width is a third of the distance between consecutive mountain folds, which results in a pleated sheet of even thickness (Figure 4(a)). Several alternatives are shown in Figure 4.

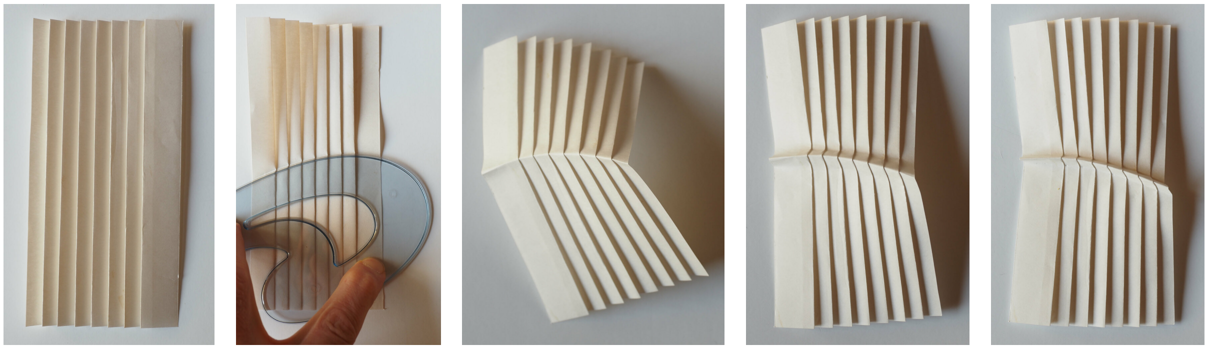


(a) *Simple pleats (knife pleats).* (b) *Double pleats (box pleats).* (c) *Tilting pleats, alternating.* (d) *Tilting pleats, alternating by two.*

Figure 4: *Four variations on folding a pleated ground. The top row shows the corresponding crease patterns, with red lines indicating mountain and blue valley folds. The first three examples have 7 pleats, based on division into 8ths; the fourth is based on 16ths.*

Folding Curves Over Pleats

Once the pleated ground is ready, the rest of the folding process is easy, even though it may sound unlikely to work until you actually try—especially if you have some experience with origami. To fold a curve, I usually flip the paper over and score the curve with a blunt tool. For complex designs, it may be useful to draw the curve first with a pencil. The score line should go through all layers, and they may have to be held flat for this. To actually fold the curve, turn the paper back over so the score line shows as the beginning of a mountain crease, and form the ridge by pinching together the two layers of this mountain crease. Depending on the pleats, there may be many more actual layers of paper here; they should not be allowed to slip too much. Pinch the fold bit by bit, moving over the curve multiple times. It helps, especially with sharply curving lines, or those that cross pleats at small angles, to not try to do too much at once. Once the mountain fold is somewhat sharp, you can work on lifting it up as far as needed, and then pressing in new valley folds at the bottom of the ridge. Again, work in small increments and move back and forth down the length of the curve. Depending on the paper, it may help to tilt the ridge back and forth, thus reinforcing the base valley folds.



(a) (b) (c) (d) (e)

Figure 5: *Folding a curved ridge over a set of pleats. After pleating (a), turn over and score the ridge curve (b). Reinforce the curve (c), determine the ridge height and slowly flatten everything else (d–e).*

Complications and Extensions (Even More Impossible Folding)

It is possible to change the height of the ridge and even reduce it until it disappears. It is also possible to merge multiple ridges into one, or have them cross at an angle. (This last feature may take additional folding and shaping.)

Figure 6: *Folding curves over crumples. Repeated crumpling and incomplete flattening shrinks the sheet. Handmade indigo-dyed washi (Awagami Factory), about 80 g/m*^{2}*, originally about* 90×60 *cm.*

As mentioned above, the ground doesn't have to be a neatly pleated sheet. In fact, linear crumpling [4] can be used to hide a sufficient amount of paper to allow us to draw curves (Figure 6).

Figure 7 shows some complications. In the red model, the lower three ridges run almost parallel, merge in the center, then continue their separate ways. What is difficult to see in the photo is that where the three ridges merge into one, the new ridge is quite a bit taller. This is intuitively necessary, because after all, the paper used up by the heights of the three ridges does need to go somewhere. Next, in the green model, one of the curves stops in the middle. This is achieved by thinning out the ridge and then trying to stop the creases just when they meet at a point. Again, without the pleated ground, this would be very difficult to achieve while leaving the surrounding area flat. (There is a whole direction in minimalist origami focused on using as few creases as possible, and this type of feature is common there [4].) It is generally easier to raise curves

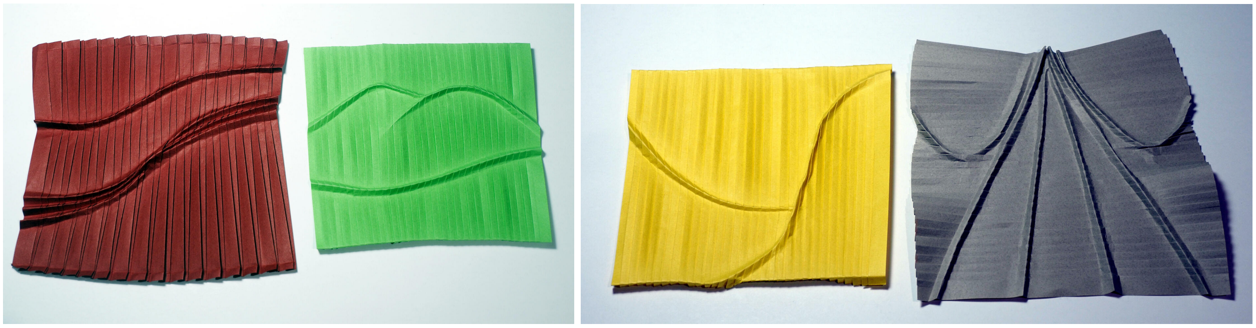


Figure 7: *Examples of added complications.*

over pleats if the angle between curve tangent and the pleats is not too small. The two rightmost examples show attempts to push this to the extreme. In the yellow model, the rightmost curve is almost parallel to the pleats, and this is difficult to fold cleanly. In the grey model on the right, an even more difficult fold is attempted: the two outermost curves change direction through a point at which they run parallel to the pleats.

Figure 8: *A pleated sheet further tessellated with curved ridges can be formed into a vessel. Pleats were tilted, shortening one end of the sheet, to form a vessel with the base narrower than the top.*

In addition to the complications shown so far, the flexibility provided by the pleats allows three-dimensional manipulation. The simplest is probably to curve the sheet into a cylinder, and then close one end, forming a vessel (Figures 8 and 9). Depending on the design, it may be worth planning where the creases hit the edges joined to form the cylinder, so that the folds can be matched and possibly interleaved to disguise the join. However, because of the flexibility provided by the folds, the forming of the vessel can be done almost without regard to the pleats and curves folded over them. In any case, glue is helpful and often necessary to hold the joined edges together.

Preparing and Leading the Workshop

I have taught this on several occasions, with groups ranging in size from a couple of people to several dozens, both in person and in a teleconference setting. No paper folding experience is needed, except if starting from a flat sheet—the initial step of folding a pleated ground for the curves is what takes the most time, and can be difficult for a total beginner. I usually avoid this problem by preparing a pleated sample sheet of paper for

Figure 9: *Three more examples of vessels using curves over pleats.*

each participant in the workshop. If using a crumpled ground instead of pleated, even this can be skipped.

Tools, paper, process

A flat surface is necessary for folding and for scoring the curves. The only tool necessary for these models is a bone (or plastic) folder, or a dried-out ball-point pen to score curves through multiple layers of paper. Most types of paper work, including basic printer paper and standard origami paper. Examples shown here were folded mostly from tant, a Japanese machine-made paper slightly heavier than printer paper. The vessel with a random field of pleats, (on the left in Figure 9) was folded from a 10g/m^2 sheet of unryu paper, and the one to its right from kyoseishi, a pre-crumpled paper of significant thickness.

To prepare the pleated sheets I usually divide into 32nds, for a total of 31 knife pleats. I find the fastest way is to use a square, about 30 cm, precrease a division into 32nds, then cut across the precreases, and pleat each half separately. (The precreasing seems to take the same amount of time whether the sheet is a 30×30 cm square or a 30×15 cm rectangle, but I can fold 15 cm long pleats more than twice as fast as the 30 cm ones.) It can also be useful to precrease some sheets so the participants can focus on pleating if time is set aside for that. Crumpling is too much fun to skip, so if that is planned, it should be done from scratch. Typically, pleating or crumpling shrinks one dimension by a factor of 2 or so.

Workshop summary

Before the workshop, prepare pleated paper samples—from a 2×1 rectangle divided into 16 or 32, fold knife pleats parallel to the short side. Optional extra: samples precreased into 16s or 32nds without pleating.

I suggest the following timing for the different parts of the workshop: 10–15 minutes: introduction—talk about the approach, show examples; 15–30 minutes: pick an example and fold it using a pleated sample; 45 minutes: free-form—options: pleat and then fold curves using precreased sheets; or fold the circle example; or crumple, then fold curves.

Circle folding

A particularly interesting exercise, if time is available, is to try and form a closed curve. In my first attempt at this, the circle in Figure 1, I used an incomplete grid based on 16×16 divisions. Figure 10 shows the arrangement of pleats. The order in which they are folded doesn't matter very much, but the one I tend to use is shown on the right in the same figure, numbering the pleats. (Four sequences of five pleats should be folded, one oriented away from each square edge, and so the first four pleats to be folded would be the four in positions labeled 1, then those labeled 2, etc.) To form the circle, it is useful to have a template in the right size, or a few of them, to share among the participants. Otherwise, the process is the same as for all of the models: divide, then pleat, then form the curve.

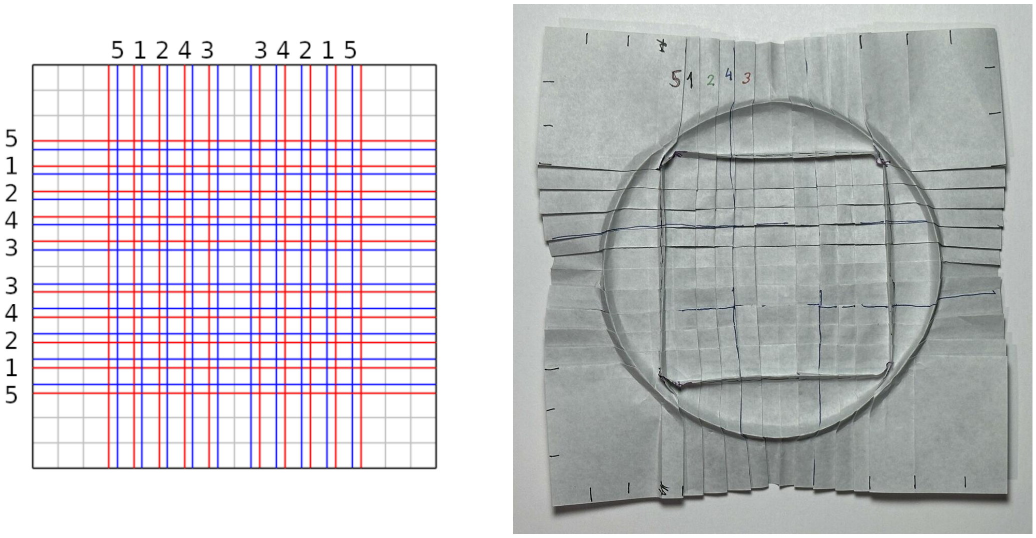


Figure 10: *Pleat pattern and order of folding for the circle. The mountain folds lie on a* 16×16 *grid.*

Acknowledgments

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