Exploring Math Illustrations: Draw, Deduce, Discuss!

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Abstract

Can you draw math? Visualizations can make mathematics concepts more accessible and engaging, but not all concepts lend themselves easily to clear illustrations. In this workshop, participants will explore how mathematics illustrations can enhance—or sometimes hinder—understanding. Through a *Pictionary*-style game, the participants will experiment with drawing and interpreting visual representations of mathematics concepts, discovering which ideas are easy to visualize and which require more creativity. The participants will collaborate, share ideas, and gain fresh perspectives on using illustration to communicate mathematics effectively.

Introduction

Mathematics concepts can be represented and visualized in many different ways. Experiencing a variety of illustrations of a concept can help one understand it from different perspectives and can provide clarity and intuition. The process of creating these illustrations can also be illuminating, even within the context of mathematical research [1]. Incorporating mathematics in artworks can provide a creative framework for exploring them and reinforce one's understanding of the underlying mathematics concepts by making them tangible. Several Bridges workshops have explored these ideas and incorporated mathematics in art forms such as poetry [3] and photography [5].

Some art forms are very well suited to illustrate specific mathematical concepts. For instance, the process of creating crochet easily lends itself to illustrating hyperbolic space [6]. Similarly, woven objects are an invitation to investigate symmetry patterns as well as combinatorial properties of the weave [2]. However, one must be very careful when choosing to illustrate mathematics through art, especially when the artwork is meant to communicate the underlying mathematics concepts. Poor design choices can lead to illustrations that confuse or mislead rather than provide insight. This workshop focuses on hand-drawn illustrations and encourages the participants to consider the question: How can we create a mathematical artwork such that the math shines through?

This workshop invites mathematics artists and enthusiasts to approach illustrations from a fresh perspective by asking how the illustrations can effectively communicate the underlying mathematical concepts. Through a series of *Pictionary*-style games, participants will discover the nuances and complexities of creating mathematics illustrations for communication. They will explore the limitations and pitfalls of common mathematics visualizations, develop a deeper understanding of how illustrations can be both illuminating and misleading, and through collaborative discussions, gain new insights and strategies for communicating mathematics visually and effectively.

Workshop Description and Outline

Pictionary is a word-guessing game in which players take turns drawing cards with prompts and making drawings based on the prompts. The other players have to guess the prompt based on the drawings within a fixed time limit. In this workshop, participants will play two versions of this game in groups of 4 to 5 players.

The first version focuses on topics that are well-illustrated while the second one focuses on those that lack well-established visual notation. Each activity will be followed by group discussions in which participants will be asked to discuss specific aspects of the illustrations made in the game. Through these games and guided discussions, the participants will discover some of the challenges of communicating mathematics through illustration. A brief outline of the workshop is as follows:

- Introductory discussion on mathematics illustrations (5 minutes)
- Math *Pictionary* Round 1: Illustrations We Know and Love(?) (25 minutes)
- Math *Pictionary* Round 2: Playing with Visual Notations (45 minutes)
- Concluding Discussion and Presentation (15 minutes)

The workshop will begin with an introduction to mathematics illustrations and how they can bring intuition and clarity to the underlying concepts. In a short phase of group discussions, the participants will be asked to share their experiences with mathematics visualizations and discuss where they have encountered them and how illustrations have helped them shape their understanding of the underlying concepts. Here, participants are already encouraged to highlight why they like or dislike certain mathematics visualizations. The participants then form groups of 4 to 5 players for the two activities. These activities are described in the following sections. At the end of the workshop, the illustrations created by the participants will be curated and shared with all the participants of the workshop.

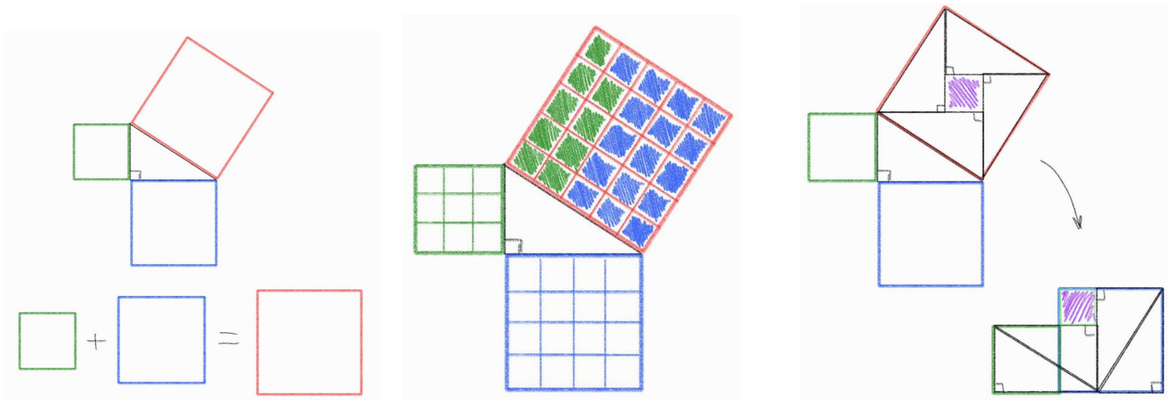
Math *Pictionary* Round 1: Illustrations We Know and Love(?)

Certain areas of mathematics, such as geometry and calculus, naturally lend themselves to visualization. They have well-established visual notations that provide a clear and consistent way to represent some mathematical ideas in these topics. While studying and exploring these topics, we come across various visual representations that we associate with the underlying mathematics. However, these visualizations may not be as illuminating as they seem. The first activity encourages participants to re-examine these visualizations critically and reflect on what they communicate and how (well) they represent the underlying mathematics.

In this activity, participants will play a version of *Pictionary* using mathematics concepts as prompts. Each group will receive a set of cards, blank sheets of paper, and colored pencils. The players will take turns drawing a card and attempting to illustrate the written prompt while the other players try to guess it. Writing is not allowed but the illustrator might be allowed to make use of numbers and symbols, depending on the prompt. Each group will time themselves and will be given two minutes per prompt to guess it correctly, counting from the first line drawn. Prompts that cannot be guessed within the time limit are counted as unsuccessful attempts.

After each player in the group has had a chance to illustrate a prompt, the group is asked to discuss the illustrations created in that round. The players are encouraged to share their perspectives on the illustrations, whether they could or could not connect them to the mathematical concept and why. The participants will be provided with a worksheet to note down their observations about each illustration. For the prompts that were guessed successfully, the discussions can focus on how the mathematics concept is represented visually. For the prompts that were not guessed, the illustrator is asked to explain their approach, and the group can dissect why the illustration was unclear or created confusion. Each participant brings their unique perspective, which can help highlight the different ways in which these visualizations can be interpreted. This activity is concluded with a brief plenary discussion. A member of each group is asked to share their experiences and insights from the activity.

Most prompts in this activity will be relatively easy to illustrate, though a few will challenge participants with concepts that lack well-established visualizations and require more creativity. The prompts will include clues to provide some context to the players as well as certain restrictions on the use of symbols or words



(a) **(b)** **(c)**

Figure 1: *Three different illustrations of Pythagoras' theorem: (a) Simple visual representation, (b)Inclusion of a numerical example, (c) A visual proof*

to make the game-play challenging and fun. For instance, in terms of context, the prompt could allow the illustrating player to reveal to the guessers that they are looking for a theorem or topological structure.

The following example prompts and illustrations show some ways that the authors envision this activity to play out. The workshop has not been tested at the time of writing this paper, however we will have test runs of the workshop before it is conducted at the Bridges Conference. The illustrations shown with the example prompts have been created by the authors.

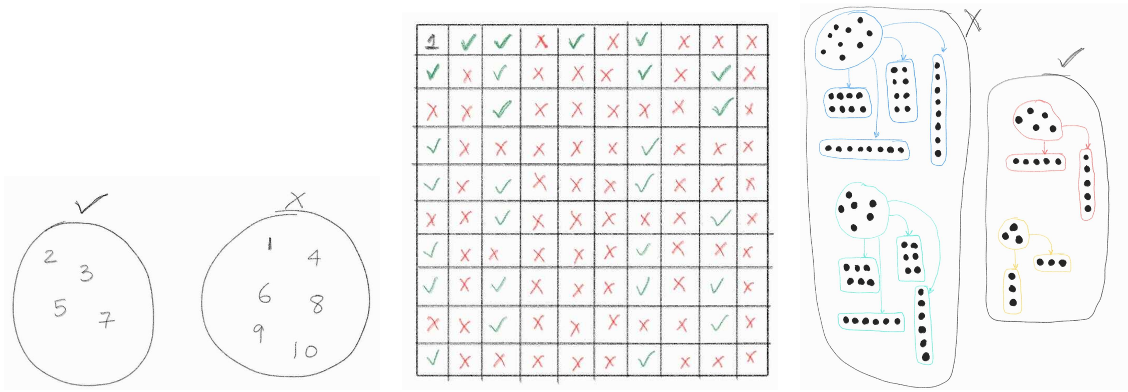
Example Prompt: Pythagoras' Theorem

Consider the prompt "In a right-angled triangle, the square of the hypotenuse side is equal to the sum of squares of the other two sides" (Pythagoras' theorem). Figure 1 shows three illustrations that a player might draw in response to this prompt. This concept is well illustrated and these example illustrations are based on various common visual representations of Pythagoras' theorem. Any of them can be associated with this prompt immediately. Geometry has well-established visual notation for depicting right angles, which can be seen in all three example illustrations. All of the examples also use color to represent distinct parts of the right-angle triangle.

The first illustration of Figure 1 states the theorem visually, but does not provide any clarity and might be confusing for someone unfamiliar with the theorem (which will likely not be the case at the Bridges conference, but is very relevant when using these illustrations in an education context when introducing the theorem). The second illustration shows a popular example of this theorem, that is, the 3-4-5 right angle triangle. The sum of the areas is represented by colored squares. While the sum is not explicitly depicted symbolically, it is inferred by the colored-in segments of the square sitting on the hypotenuse of the triangle. This example therefore includes a numerical example. The third illustration shows a way of dissecting the square of the hypotenuse and rearranging the pieces to show that the area is equal to the sum of the areas of the squares of the other two sides. Although this can be considered a visual proof of the theorem and can communicate more about the theorem than the first two illustrations, this may not be immediately clear from the image.

Example Prompt: Prime Numbers

A trickier prompt in this round is "Prime numbers", since this is a concept that is not usually explained visually. The difficulty of this prompt depends on whether the illustrator is allowed to write numbers in their illustration. Figure 2 includes three possible illustrations for this prompt. The first illustration depicts



(a) **(b)** **(c)**

Figure 2: *Three different illustrations of prime numbers: (a) Using digits, (b) Sieve of Eratosthenes, (c) Factors by dot arrangement.*

two sets of numbers. Note that *Pictionary* has a temporal component: The guessing members of the group can already start guessing while the illustrating person is drawing. By adding numbers in sequence to the appropriate set, or even by simply writing a list of prime numbers, the prompt can be guessed easily by those who are familiar with prime numbers. This is comparable to the first illustration in Figure 1, which also states the prompt and expects the guessing players to rely on their existing knowledge to guess the prompt correctly.

The second illustration of Figure 2 shows a visual representation of the Sieve of Eratosthenes. This is a procedure to determine which numbers in a certain interval is prime. The algorithm is executed by marking the next non-marked number as prime—starting with 2—, crossing out all its multiples in the chosen interval, and moving on to the next non-marked number. This guarantees that by the end of the procedure, only prime numbers are marked. By filling in the squares in the right sequence, the illustrator can visually explain this algorithm in a way that can be understood even by those who are not already familiar with it. In this case, the concept is communicated not just through the image, but the sequence used to draw the image. Furthermore, it does not only illustrate the concept of a prime number, but also an algorithm to delineate prime from composite numbers.

The last illustration of Figure 2, like the first one, shows two sets of numbers. The numbers are depicted as sets of dots instead of writing them symbolically. The factors of each number are shown by arranging the dots representing the number in rectangles. We can see that the dots representing prime numbers can only be arranged in a vertical or horizontal line, illustrating the fact that their only factors are 1 and the number itself. Although this illustration encapsulates the definition of a prime number, the unfamiliar and unconventional visual notation might create confusion.

The examples from Figures 1 and 2 show some of the variety of illustrations that we can expect from the game. Also, we have hinted at the kind of discussions that they might spark. The prompts that can be visualized easily and have popular visualizations, such as those shown for Pythagoras' theorem in Figure 1, can encourage participants to question the clarity and efficacy of these illustrations. The trickier prompts are likely to result in the use of creative and unconventional visual notation, which might result in a wider variety of illustrations across groups.

Through this activity, the participants are encouraged to re-examine their ideas about visualizations that they have taken for granted. The group discussions can lead to insights on different ways in which these well-known illustrations can be interpreted. Examining the difficulty of various prompts will help bring out the specific challenges of creating illustrations for concepts that lack well-established visual notation.

Math *Pictionary* Round 2: Playing with Visual Notations

Certain mathematical topics, particularly more abstract ones like algebra and number theory, are harder to visualize. In the first activity, participants tackle a few (comparably easy) prompts from such topics. Through these prompts, we highlight how these concepts differ from those that can be illustrated using well-established visual notation and explore some of the ways they can be illustrated. The second activity—another game of Math *Pictionary* with some twists—builds on this by challenging participants to collaboratively develop meaningful visualizations for series of prompts from topics that are not commonly visualized.

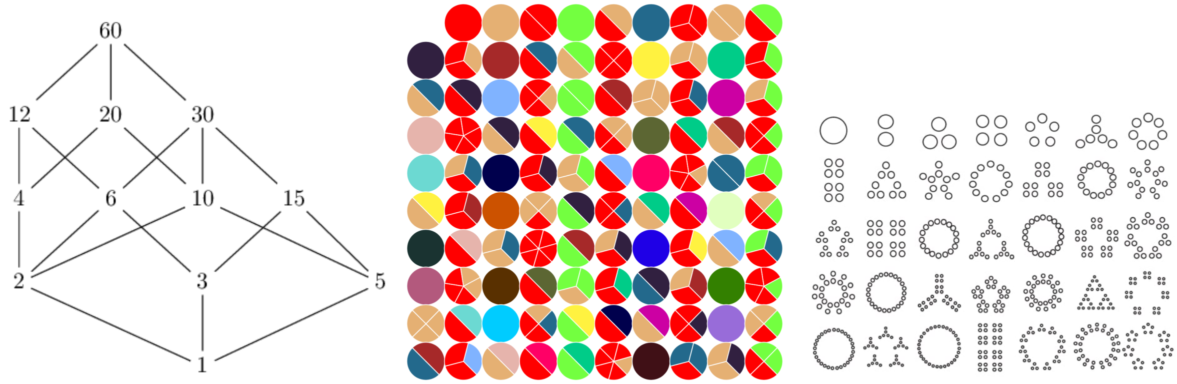
In this round, each group selects a mathematical topic to illustrate. To support meaningful participation from all members, we offer a wide range of options so that each group can select a topic that all members of the group are comfortable with. Based on their selection, each group is given a set of example illustrations for some concepts in their chosen topic, along with descriptions that explain how these illustrations show the underlying concepts. Group members are asked to discuss these example illustrations to develop a clear understanding of how they represent the underlying concepts. The visual notation used in the example illustrations can be used as inspiration for illustrating prompts in this activity.

As before, each group receives a set of cards with prompts. The prompts are mathematical concepts related to their chosen topic that the participants are asked to illustrate. However, unlike the previous activity, the prompts are in a fixed sequence and the concepts build on each other and increase in complexity. The game starts with a player illustrating the first prompt. As before, the other group members are given two minutes to guess the prompt. The illustrator is encouraged to use the visual notation used in one (or more) of the example illustrations, but they may also choose to use a different visual notation if they think it is more suitable. After the prompt has been revealed to the group, the players discuss the efficacy of the illustration, this time focusing specifically on the choice of visual notation. The other players may suggest tweaks to the illustration to improve clarity, as well as alternative illustrations that use a different visual notation. The group is encouraged to discuss which notation is most suitable for the prompt and why.

The next illustrator is encouraged to build on one or more of the previous illustrations and borrow visual notation wherever possible. When an illustrator suggests a new or modified visual notation, the group is asked to reconsider old prompts to see if any of them can be illustrated effectively using the new notation. Through this process of experimenting with visual notations, the group creates a set of illustrations for each prompt. After each member has illustrated a prompt, the group is asked to discuss which visual notations worked best for which concepts and why. They are asked to organize the visualizations made in the activity by visual notation and write brief explanations of the visual elements used in the notations and what they represent. At the end of the activity, the visualizations created in the game are presented and all the participants have the opportunity to look at the illustrations created by other groups.

Example: Number Theory

One of the topics offered for this activity is number theory. In this case, the prompts can include concepts like multiplication, division, the modulo operation, factors of a number, coprime numbers, prime factors, least common multiple (LCM), greatest common divisor (GCD), and the Euclidean algorithm for calculating the GCD. Figure 3 shows some examples of visualizations of factors of a number. These are the kind of example illustrations that will be provided along with the prompts. These three visualizations show three distinct ways to depict numbers and provide different information about their factors. The first image shows a Hasse diagram of the factors of the number 60. The second image shows a visualization in which every number is depicted by a circle with one or more colors. Circles representing prime numbers have exactly one color, and other numbers have colors representing the prime factorization of that number. The third image depicts each number as a set of circles and the arrangement of these circles provides information about the factors of the number.



(a) **(b)** **(c)**

Figure 3: *Three inspirational illustrations for factors: (a) Hasse Diagram for the number 60, (b) Prime Factorization Diagram [4], (c) Brent Yorgey's Factorization Diagrams [7].*

The participants can use the visual notations used in these examples (or some combination of them) in their illustrations. By adapting notation used in these examples, they can be certain that their group members will be able to identify the visual elements they are using in their illustration and relate them to the appropriate concept. Figure 4 shows an example of how the prompt "Division" can be illustrated using visual notation similar to that of the last illustration in Figure 3. Numbers are visualized as sets of encircled dots, similar to the third example illustration. Distinct colors are also used to show distinct numbers. The process of dividing the number is shown by rearranging the dots. This illustration can easily be modified for prompts like "Quotient" and "Remainder". In this example, the quotient is highlighted by encircling a row of dots in the rectangle.

Compare this to the illustrations in Figure 5 that attempt to illustrate the same prompt—"Division"—but this time using modifications of the visual notation used in Hasse Diagrams. In the first illustration in Figure 5, lines between numbers represents addition rather than factors. Although this illustration appears to use the Hasse diagram notation, it instead changes the meaning of the visual notation. This can create a lot of confusion, especially for those familiar with the Hasse diagram notation. In this game of Math *Pictionary*, this would be a very poor choice of notation. The second illustration in Figure 5 attempts to circumvent this issue by creating a new notation of a dotted line to represent addition. However, both illustrations are rather poor representations of the given prompt and we can easily see that the illustration in Figure 4 is better at illustrating the concept of "Division".

However, this cannot be assumed for all concepts in this set of prompts. Consider the prompt "Co-prime numbers". Co-prime numbers are pairs of numbers that have only 1 as the common factor. Figure 6 shows two examples of how this prompt can be visualized—one that uses the dot notation and one that uses the

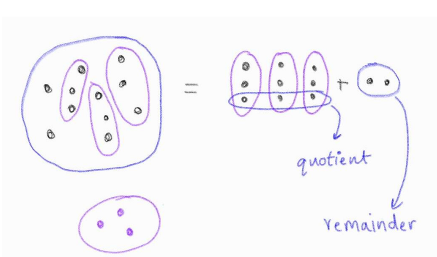
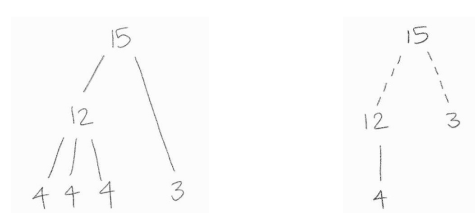
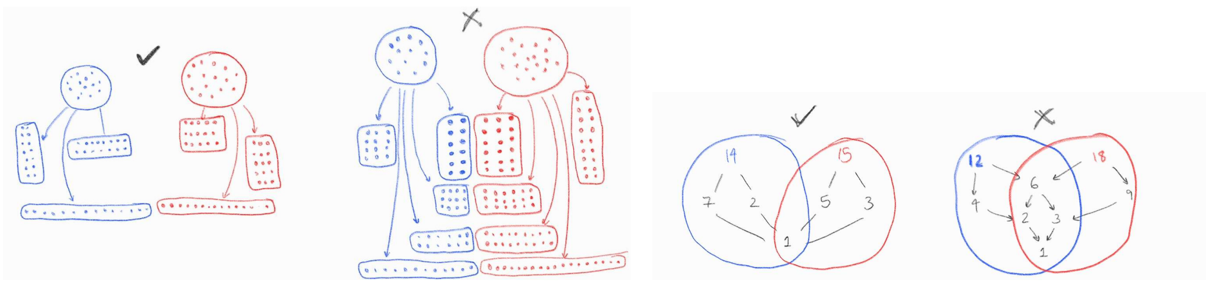


Figure 4: *Division*



(a) **(b)**

Figure 5: *Illustrating division using Hasse diagrams*



(a) **(b)**

Figure 6: *Two illustrations of the concept of co-prime numbers: (a) Dot notations, (b) Hasse diagrams*

Hasse diagram notation. Both illustrations show two examples of two pairs of numbers with their factors. In both cases, the co-prime pair is highlighted using a check mark, while the other pair is marked with a cross. The first illustration, in which the factors of the numbers are shown by arranging the dots in rectangles, does this by placing rectangles of the same height—and hence representing common factors–next to each other. The co-prime pair has only one such pair of rectangles, while the other pair has many.

The second illustration of Figure 6 uses a modification of the Hasse diagram notation. Here, the two Hasse diagrams overlap at the common factors. This is further highlighted by drawing colored circles around the Hasse diagrams of each number, with the circles intersecting exactly at the common factors. The co-prime pair has exactly the number 1 as the common factor, whereas the other pair has many common factors. In this case, the illustration that uses the Hasse diagram notation is much clearer, since the illustration that uses the dots is quite cluttered and the corresponding pairs of rectangles are hard to identify.

These examples highlight how the participants might experiment with visual notations in this activity. Through this process of creating good and bad illustrations, they will explore the possibilities and limitations of various visual notations. The collaborative nature of this activity encourages the participants to explain their thought process, which helps them identify biases and assumptions.

Through this activity, the participants will explore a variety of visual notations that they can use to visualize concepts in their chosen topic. This process of exploring the possibilities and limitations of various visual notations for a certain topic can lead to new strategies for representing mathematical concepts as well as gain new insights into how visual elements contribute to an illustration. The participants will also develop a deeper understanding of why different visual notations are suitable for different concepts and how to make smart choices of visual notations when creating illustrations.

Adapting the Workshop for Different Audiences

This workshop is designed for the audience at Bridges, and to explore the nuances of creating mathematics illustrations for communication. The prompts have been designed to include topics that are at a high-school level and the discussions focus on the ability to represent and communicate the underlying mathematics using illustrations. Specifically the second activity is designed to focus on the unique challenges of creating mathematics illustrations. However, the first activity used in this workshop can easily be modified for different audiences and objectives by changing the prompts and the focus of the discussions. Such activities can also be used as team building exercises and can serve as icebreakers that encourage participants to share their ideas and work collaboratively.

A modified version of the first activity can be used in classrooms as a fun game to review a topic after it has been taught. Instead of working in small groups, the students can draw their illustrations in front of the class and the teacher can guide the following discussions to reinforce concepts and address misconceptions.

For topics in geometry where a lot of new visual notations are introduced, the activity can be used to check whether students have correctly and completely understood the notation being used in the chapter. Another simple modification is to change the mode of communication. Students can be asked to use clay, paper, or other materials to represent concepts instead of drawing illustrations. They can also be asked to enact the chosen concept individually and in groups. This can encourage students to play with mathematics and creatively explore different perspectives on the concepts they are learning.

Another version of this workshop can include concepts with more advanced topics in mathematics and can encourage researchers to reconsider how they communicate their work. Participants can be given time to research their concept and visualize it and think of interesting ways to illustrate and explain it. It might be necessary to increase the time limit provided for guessing the prompts. The discussions can focus on the mathematical concepts rather than the illustrations.

Conclusion

This workshop encourages participants to reflect on what visual notation makes for an effective mathematical illustration. The first activity focuses on familiar images we associate with mathematics concepts—usually ones that are commonly used to explain these concepts—and asks participants to reconsider their efficacy. The second activity takes a different approach and challenges participants to think about how visual elements and notations contribute to illustrations by experimenting with visual notations for topics that are not commonly visualized. The *Pictionary*-style games make the illustration process playful and engaging and encourage participants to share their ideas and perspectives and work collaboratively. These games challenge participants to think about mathematics concepts from different perspectives and communicate their ideas creatively. By changing the prompts, guided discussions, and mode of communication, the workshop can be adapted to suit a variety of audiences with different learning objectives. We hope participants leave the workshop with a fresh perspective on mathematical illustrations and feel inspired to create art where the mathematics truly shines through.

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