Fold-and-Cut Lines for the Hat, Turtle, and Spectre Tiles

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Abstract

We present crease patterns for flat-folding a piece of paper and producing the hat, turtle, or spectre tile with one cut.

Introduction

In 1998, Demaine, Demaine, and Lubiw proved the Fold-and-Cut Theorem [2]. It states that any graph made from straight line segments in the plane—connected or not—can be cut from a piece of flat-folded paper by making a single straight cut. The proof had some gaps, but Demaine, Bern, Eppstein, and Hayes successfully proved the result not long afterward using a different method [1]. See [3] for more details.

In this article, we show how to produce the hat, turtle, and spectre tiles [4, 5]—the aperiodic monotiles—with one cut. Figure 1 shows the hat tile example.

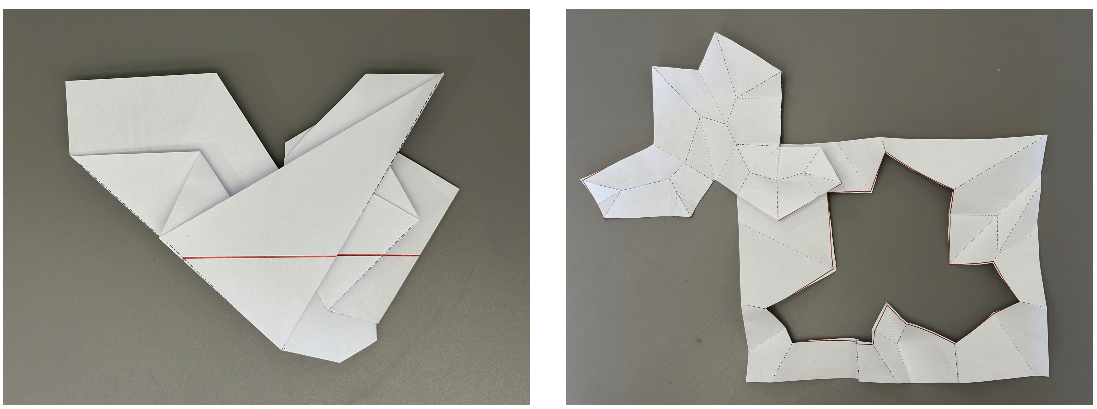
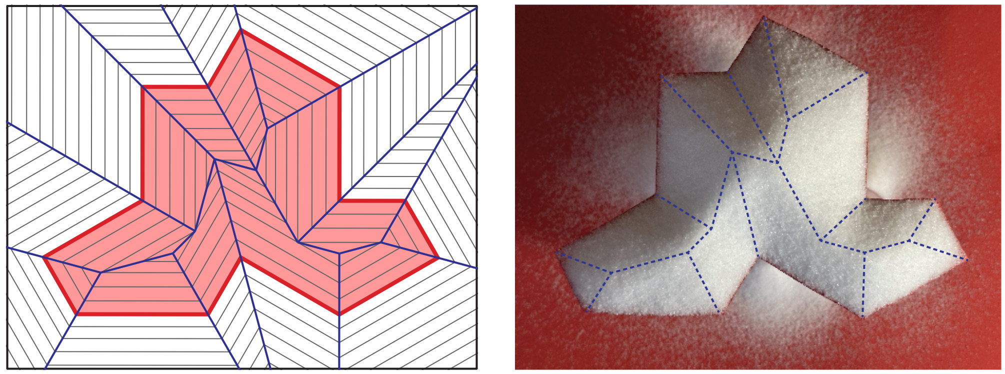


Figure 1: *This paper is folded so that a single cut along the red line produces the hat tile.*

The Hat Tile

This section describes how we created the crease pattern for the hat tile using a procedure that also works for the turtle. The spectre is a little different, which we describe later. We used the algorithm described in [2], which does not always work but does for these shapes. The construction is based on two elementary facts. First, folding the paper along the bisector of an angle will bring the two line segments together. Second, folding the paper along a line perpendicular to a segment will fold the segment on top of itself.

Instead of simply drawing angle bisectors at each vertex, we produce the *straight skeleton* of the polygon. Imagine there is a pen at each vertex. Uniformly shrink the polygon with each edge moving inward. The pens draw the angle bisectors. When two bisectors meet, a side of the polygon disappears and the polygon may become disconnected. Continue the process with the new polygon. Afterward, repeat the process by enlarging the polygon. This process produces the straight skeleton graph in blue in Figure 2a. Interestingly,

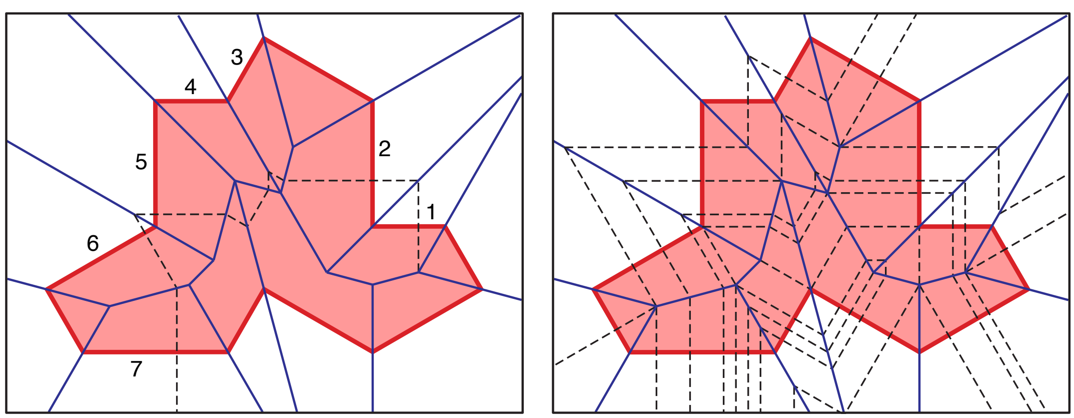


(a) (b)

Figure 2: *(a) The blue lines form the straight skeleton of the hat tile. (b) The ridges along the salt mountain form the straight skeleton.*

the interior straight skeleton is also the collection of ridgelines when salt is poured on the raised polygon (see Figure 2b). The straight skeleton is a good start to producing the crease pattern, but we need more fold lines.

The next step is to draw the *perpendiculars*—segments perpendicular to the shape's edges. The straight skeleton divides the paper into regions, each containing one edge of the shape. From each vertex of the straight skeleton, draw, if possible, a segment perpendicular to the side in each neighboring region. Such a perpendicular may reach the edge of the paper, a straight skeleton vertex, or a straight skeleton edge. In the latter case, it enters another region, so it heads in a direction perpendicular to the hat's edge in that region. Figure 3a shows one that starts at a vertex and is perpendicular to edge 1. When it reaches the straight skeleton, it proceeds perpendicular to the hat's edges 2, 3, 4, 5, 6, and 7. Figure 3b shows the perpendiculars.

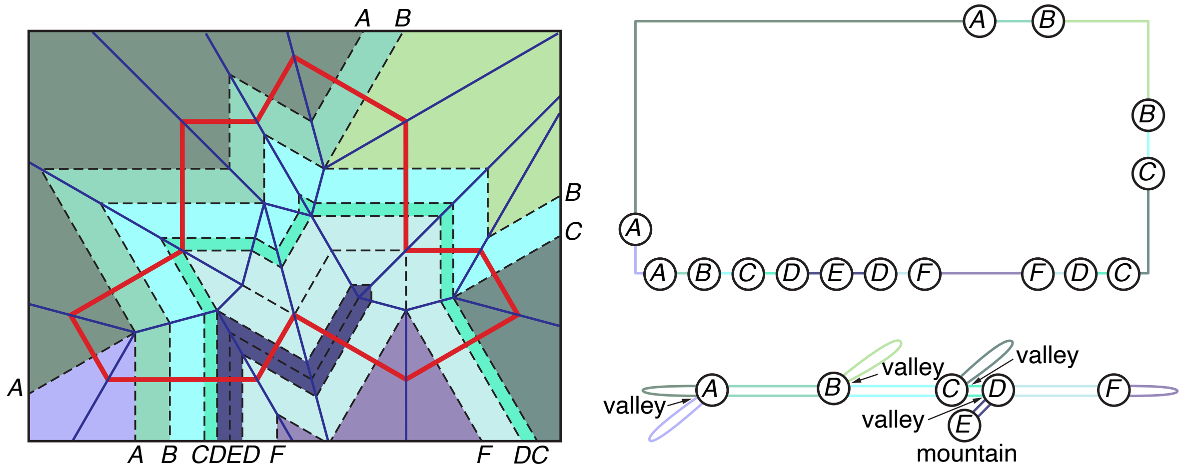


(a) (b)

Figure 3: *(a) One perpendicular for the hat tile. (b) All perpendiculars (right)*

The final step is determining which lines become fold lines and whether they should be mountain or valley folds. First, the straight skeleton: all these lines are fold lines. Each segment in the straight skeleton is the angle bisector of an angle during the construction of the straight skeleton. Those from acute angles (or from parallel lines) become mountain folds, and the rest become valley folds.

Next, we turn to the perpendiculars. Label the locations where the perpendiculars reach the edge of the paper so that any points connected by perpendiculars have the same label. Figure 4a shows the points labeled A to F. Suppose a string ran around the edge of the paper with the points A through F labeled. When we bring together the points with the same label, we obtain a graph with doubled edges, such as in Figure 4b. This string-graph folds flat by choosing to make a valley fold, a mountain fold, or no fold at each node.



(a) **(b)**

Figure 4: *(a) The perpendiculars divide the paper into regions. (b) The boundary can be folded flat when we bring the points with the same label together.*

The perpendiculars divide the paper into the colored regions in Figure 4a. Cutting out and folding a region along the straight skeleton would yield a folded strip whose edges are perpendiculars or a folded stack with an edge consisting of perpendiculars. If we didn't cut out the regions before folding, they would look like the graph in Figure 4b when viewed edge-on. The mountain and valley folds in the string-graph determine the mountain or valley folds for the perpendiculars where they meet the edge of the paper. Moving away from the edge, a perpendicular alternates mountain and valley each time it crosses the straight skeleton.

Figure 5 shows the resulting fold lines for the hat tile and for the turtle tile, which can be found similarly.

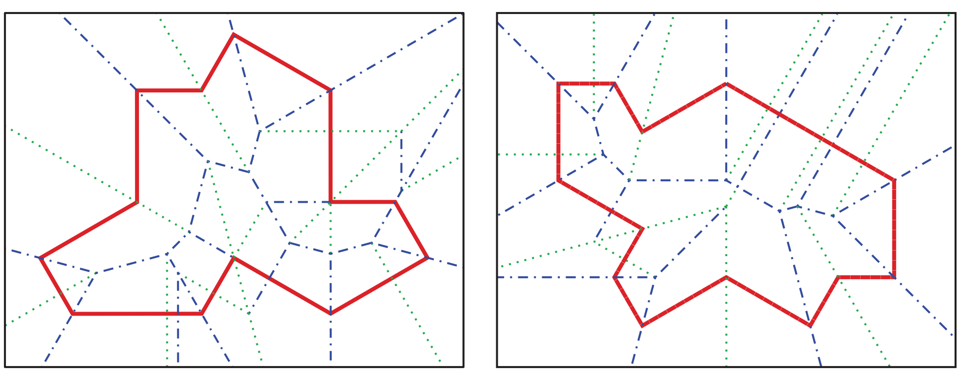
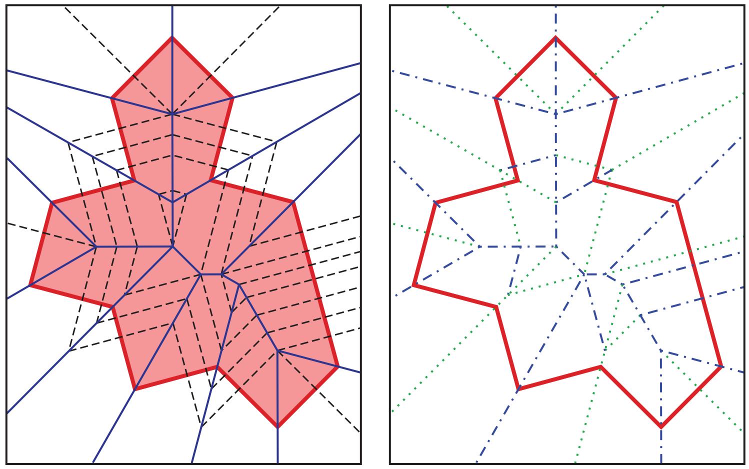


Figure 5: *The fold-and-cut crease patterns for the hat and turtle tiles.*

The Spectre

The method for obtaining the crease pattern for the spectre tile is more complicated than for the hat and turtle. (Note: Technically, the tile we are calling the spectre is called $T(1, 1)$ in [5], and a spectre is a strictly chiral aperiodic monotile obtained from $T(1, 1)$.) In the hat and turtle constructions, all regions bounded by perpendiculars extend to the edge of the paper. However, the spectre tile has circular corridors—regions completely contained in the paper, as shown in Figure 6a. In this case, the algorithm in [2] does not apply. Moreover, while theoretically accurate, the technique in [1] is less practical for producing fold patterns.



(a) **(b)**

Figure 6: *(a) The straight skeleton for the spectre tile (solid lines) and all perpendiculars (dashed). (b) The fold-and-cut templates for the spectre tile.*

The given algorithm got us close to the crease pattern for the spectre. We then used ad hoc methods to find the mountain and valley assignments in Figure 6b. Our methods were not simply trial and error. We need our paper folded flat, and this imposes certain constraints on the crease pattern not present in three-dimensional origami. The following constraints can be helpful when trying to obtain the crease pattern: The Maekawa–Justin theorem states that the numbers of mountain and valley folds at any vertex must differ by two. So, each vertex must have an even number of folds, and it can't have, for instance, two mountain and two valley folds. The Kawasaki–Justin theorem states that a vertex is flat-foldable if and only if the alternating sum of the angles is zero. Lastly, if a fold angle is less than its neighbors, it must be bounded by one mountain and one valley fold. See [3, ch. 12].

Printable versions of these designs are available as supplemental material to this paper. The dot-dash lines are mountain folds, the dotted lines are valley folds, and the red lines should align and be the single cut line. Folding the paper is tricky. Rather than doing one fold at a time, the algorithm expects the folding to happen "all at once." So, it is best to make all the creases first and then fold the paper flat, little by little.

References

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