Magnetic Generalized Residue Designs

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Abstract

In this article, we explore the multiplicative structure of the set of integers modulo N via string design. Furthermore, we introduce novel visualizations of these structures using induced magnetic fields corresponding to string designs.

Introduction

String art has a long history as both an artistic medium and a tool for exploring mathematical concepts. One such concepts is the structure of integers modulo N. Let $N > 0$ be an integer. Consider a circle C whose circumference is divided into N equal arcs. We refer to the points that separate adjacent arcs as vertices. Label the vertices by $0, 1, 2, \cdots, N-1$. Fix a positive integer a in $\{1, \cdots, N-1\}$. An (N, a) *residue design* [2, 3] is obtained by drawing a straight edge between each vertex i and its a multiple ia, modulo N. This construction — also known as modular multiplication table — is well-known due to the interesting designs it creates including the "cardioid" and more generally "the n-cusp epicycloid".

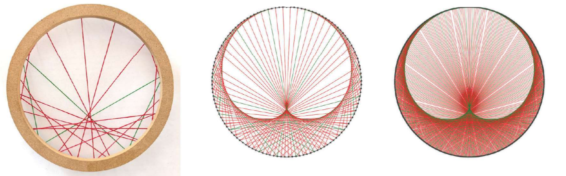
We used the idea behind residue designs to give a visual exploration of multiplicative structure in $\mathbb{Z}_N$ via string art. We also introduce the technique of replacing strings with wires and explore the imagery generated by the induced magnetic field when current passes through wires. This idea initially was an attempt towards fully reflecting the multiplicative structure of $\mathbb{Z}_N$ by assigning *weights* on the strings. We achieved imposing a binary weight by controlling the direction of the current passing through the wires. The imagery resulting from visualizing the magnetic filed induced by the string designs is fascinating from visual point of view and, in some instance, informative about the underlying mathematics. One of our original motivations for this work was to explore the possibility of physical realizations of these fields using iron filings suspended above custom-built wire arrays. That goal is not fully realized yet and the current paper showcases the magnetic residue designs through modeling the magnetic field.

Generalized Residue Designs

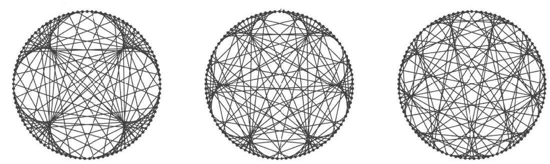
In this section, we assume the reader is familiar with undergraduate abstract algebra and number theory at the level found in [1]. Let U_N denote the group of multiplicatively invertible elements or units in $\mathbb{Z}_N$. These are precisely those integers in $\mathbb{Z}_N$ co-prime to N. Let's pick a in $\mathbb{Z}_N$ such that $\gcd(a, N) = 1$. This will guarantee that a is a unit modulo N and hence is in U_N. There are $\phi(N)$ elements in U_N, where ϕ is the Euler's totient function. Hence, there are $\phi(N)$ possible a to choose. Construct the following sequence

$$1 \to a \to a^2 \to a^3 \to \cdots \to a^r = 1 \tag{1}$$

All the computations are modulo N. This way each term corresponds to a vertex on the circle C. We continue the process until we get $a^r = 1$. Connect the vertices labeled by consequent terms of the above sequence via a straight edge. The result will be a design to which we refer as a *Generalized Residue Design* (see Figure 1). Note that the order r of a is a divisor of the order of U_N and hence finite. This guarantees that the sequence ends and we come back where we started. The terms in the sequence (1) comprise the subgroup $\langle a \rangle$ generated by a in U_N. Hence the Generalized Residue Design depicts a subgroup in U_N. If U_N is cyclic and the order of a is $\phi(N)$, then the sequence (1) will cover all the vertices in U_N. By the Primitive Root Theorem, this can happen only if $N = 1, 2, 4$, p^k or $2p^k$, for an odd prime p and a positive integer k. The significance of this special case becomes more apparent when one attempts to create the design with a string. If a generates U_N, the design can be generated with a single string that passes through all the vertices labelled with elements of U_N. All the other vertices are zero divisors. Figure 1a shows this design for $N = 5^2, 5^3$ and 5^4 with $a = 2$. Here $\langle a \rangle$ is constructed by the red string. The green shows the sequence generated by the zero divisor 5.



(a) *$N = 5^2$, 5^3 and 5^4 with $a = 2$.*



(b) *$N = 97$, $a = 5$, $a = 7$, and $a = 10$.*

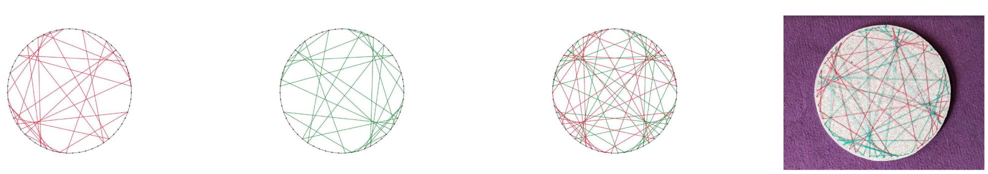
Figure 1: *Generalized residue designs.*

A rather special case of cyclic U_N is the design obtained when N is a prime number and a is a primitive root mod N. In this case, our *generalized residue design* overlaps with the *residue design* discussed in [2, 3]. In general, once we have a generator a for a cyclic group of order M, we can find other generators. Indeed, a^j also generates $\langle a \rangle$ if and only if $gcd(M, j) = 1$. Interestingly, the designs induced by different generators are different. That is, the residue design picks up information about the pair $(a, \langle a \rangle)$ as opposed to just $\langle a \rangle$. For example, 5, 7 and 10 are all generators for U_{97} generating different designs as showed in Figure 1b.

Even if a generates a proper subgroup of U_N, the designs are aesthetically rich, especially when $\langle a \rangle$ in U_N has a small index. That means $\frac{\phi(N)}{r}$ is small, where r is the order of a. Let us further explore the case where a generates a small index subgroup. Let's think about $N = 59$ and $a = 5$. The order of 5 in U_{59} is 29. Note that $\phi(59) = 58$ and hence $\langle 5 \rangle$ in U_{59} has index 2. Figure 2a illustrates $\langle 5 \rangle$. The rest of the vertices not covered in $\langle 5 \rangle$ are in the only other coset. We can generate a design that covers the other coset by picking any vertex $b \notin \langle 5 \rangle$, and construct the sequence

$$b \to ba \to ba^2 \to ba^3 \to \cdots \to ba^r = b \qquad (2)$$

In our example, we can take $b = 2$, and generate a design via sequence (2). Figure 2b illustrates the coset $2\langle 5 \rangle$. By Lagrange's theorem, the cosets of $\langle a \rangle$ partition U_N. So superimposing them, we get a design that covers all the vertices as shown in Figure 2c. To generate this coset decomposition with strings, we will need two strings, one for each coset as shown in Figure 2d.



(a) **(b)** **(c)** **(d)**

Figure 2: *Coset partition of $\langle a \rangle$ in U_N for $N = 59$ and $a = 5$, (a) the subgroup $\langle 5 \rangle$, (b) the coset $2\langle 5 \rangle$, (c) cosets partition U_{59} (d) string art based on the coset partition.*

The design in Figure 2c and 2d can also be obtained through the multiplication table $(59, 5)$. However, that method does not reveal much about the number of cosets of $\langle 5 \rangle$ in U_{59}. Unless we use an external identifier to keep track of the coset structure, the coset information flows in the process only and not in the final design. For instance, the identifier could be the color of the strings chosen in Figure 2. In the next section, we replace the strings with wires and use the direction of electric current passing through wires to identify the cosets.

Electromagnetic Field Analysis

The Biot-Savart Law, by Jean-Baptiste Biot and Félix Savart, quantitatively calculates the magnetic field B that an electric current I flowing through a wire generates at the position $\mathbf{r}$ in the 3D space, linking the electric current's geometry and magnitude to the magnetic field's characteristics $B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I\, d\boldsymbol{\ell} \times \mathbf{r}'}{|\mathbf{r}'|^3}$.

The Biot-Savart Law, crucial for deriving the magnetic field (B) around a conductor with steady current (I), employs a line integral over the conductor's path. Considering the differential element ($d\vec{\ell}$) and its position relative to the observation point, this law integrates the contributions from each segment, using the magnetic constant (μ_0), to relate electric current geometry to the magnetic field's distribution, emphasizing the impact of current flow on field creation. To apply the Biot-Savart Law, we utilize line integration across the configurations of wires in our generalized residue design. This method involves integrating over the current-carrying wire segments to compute the magnetic field at various points in space by producing a 3D mesh around the wires. Through this method, we observe the electromagnetic patterns emerging from different residue design scenarios. Aesthetically speaking, this methods gives a novel way of visualizing multiplicative group U_N. To showcase this, we generated the field induced by the design depicted Figure 1b, where we generated U_{97} with three different generators 5, 7, and 10. This is shown in Figure 3. Each row represent the quiver (left) and the contour (right) of the induced field at a distances d from the design.

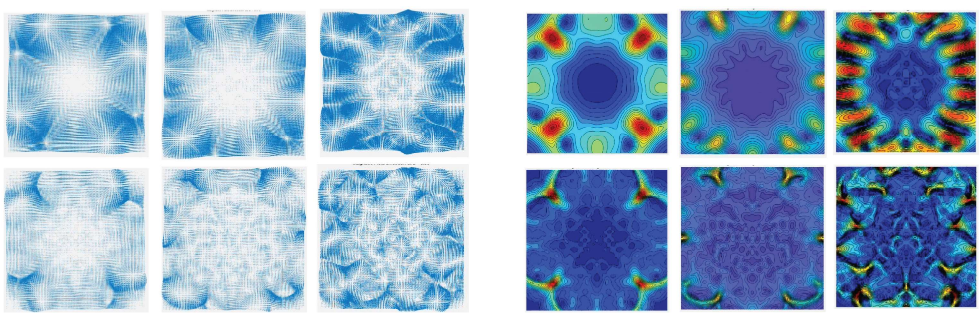


Figure 3: *Quiver and contour induced by U_{97} as generated by 5, 7, and 10 at two distances d_1 and d_2.*

We would like to explore whether this new visualization holds visual information connected to the patterns generated by the number of cosets inside U_N. To explore, we simulated the magnetic field induced by the design obtained from $\langle 5 \rangle$ in U_{59}, and its coset partitioning of U_{59} and plotted the quiver and contour of the field generated on a plane parallel to the planar design located at distance d from it in Figure 4. The first column in quiver and contour shows $\langle 5 \rangle$ in U_{59}. We explored the effect of the direction of the current in identifying the different cosets. First, we tried to run a current with the same direction through both $\langle 5 \rangle$ and $2\langle 5 \rangle$ (the second column in the quiver and contour in Figure 4), and then we switched the direction of the current between the two cosets (the third column in the quiver and contour in Figure 4). We were able to generate a variety of patterns as the distance d from the design changes. Rows in Figure 4 correspond to a reasonably close distance d_1 and a farther distance d_2. We noticed that when using the opposite direction for the current, the resulting pattern creates a duality that seems to be connected to the index of $\langle 5 \rangle$ in U_{59}.

Motivated by the promising patterns for $U_{59}/\langle 5 \rangle$, we decided to explore another design that admits three distinct cosets: cosets of $\langle 7 \rangle$ in U_{73}. To make the trivial coset, we generated the sequence (1) with $a = 7$. Then to generate the second coset, we chose b_1 to be the smallest label on the circle that is not in

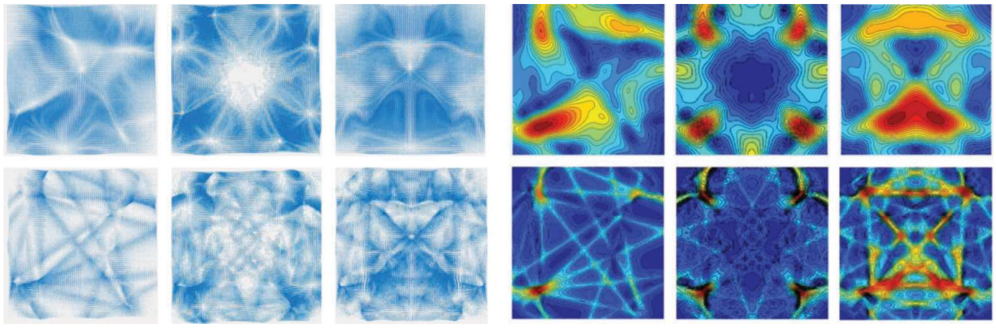


Figure 4: *Quiver (left) and contour (right) induced by the design $\langle 5 \rangle$ in U_{59} at d_1 and d_2.*

$\{a, a^2, a^3, \cdots, a^r = 1\}$, and generate sequence (2) with $b = b_1$. Note that we are making a specific choice here that affects the design. We repeat the process by choosing the next value for b to be the smallest label not covered in $\{a, a^2, a^3, \cdots, a^r = 1\} \cup \{ba, ba^2, ba^3, \cdots, ba^r = b\}$. A current is labeled with +1 if it follows the direction of sequence (1) or (2), and -1 if it is the opposite direction. We tried the following labeling of the cosets from left to right: $(1, 1, 1)$, $(1, -1, 1)$ and $(-1, -1, 1)$. This is shown in Figure 5, the left most column is $\langle 7 \rangle$ followed by the aforementioned labeling of the coset decomposition. Rows correspond to a reasonably close distance d_1 and a further distance d_2. The index seems most apparent in $(-1, -1, 1)$ labeling. To complement the static plots presented in this work, we include a series of supplementary videos that show the magnetic field evolving in a plane parallel to the wire configuration at increasing distances d.

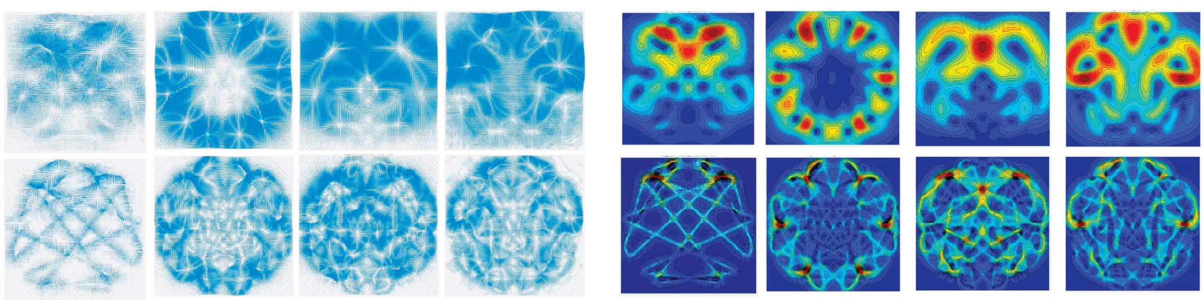


Figure 5: *Quiver (left) and contour (right) induced by the design $\langle 7 \rangle$ in U_{73} at d_1 and d_2.*

Conclusion

The electromagnetic field analysis gives a novel method of visualizing the multiplicative structure of $\mathbb{Z}_N$. In the examples we explored, the index of the subgroup generated by our choice of the generator seem to be visible in the pasterns. Beyond the mathematical significance, these visual construction carry a spatial rhythm, resembling the layered symmetry and curvature found in physical string art. Moreover, the construction offers a natural three dimensional extension of the designs via the magnetic fields they generate.

References

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