Six Platonic Bubbles, or Regular Bubble Shapes Conforming to Joseph Plateau's Rules, 1873

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Abstract

This paper deals with the six regular bubble shapes that are possible conforming to Joseph Plateau's 1873 laws defining soap film configurations.

Introduction

Joseph Plateau (1801–1883), a Belgian physicist and mathematician, published in 1873 a massive treatise about his experimental and theoretical work on surface tension and minimal surfaces [4]. Plateau describes his experiments with soap bubbles and soap films, formed on variously shaped wire frames dipped in soap liquid, and recounts his own fascination with the beautiful surfaces emerging.

Plateau formulated four rules for soap films, known as Plateau's Laws. Two of these define the geometry of soap film borders:

1. Only three film surfaces can meet at a border, and they always meet at a fixed angle of 120° = arccos(–1/2);
2. A vertex always constitutes a meeting of four borders, that is six faces forming four three-faced vertices. And the corner angle of each face in the vertex equals the tetrahedral angle arccos(–1/3) ≈ 109.5°.

Plateau's Laws are no fundamental Laws of Nature; they merely describe the shaping of soap films due to surface tension (Figure 1). The concepts of *border* and *edge* both refer to a line where soap films meet: border being a meeting line of three film surfaces, whereas edge, in reference to bubbles, points to the border being a meeting line of two faces of the bubble.

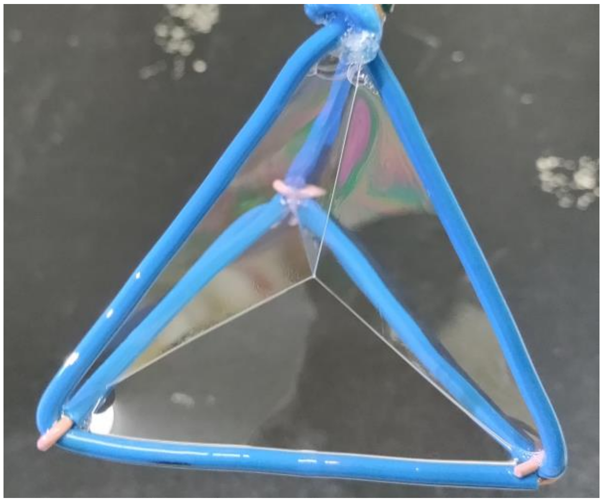


Figure 1: *The geometry of soap films in a tetrahedral frame. Three surfaces meet at each of the four borders; these form a fourfold vertex in the tetrahedral center.*

Personally, I shared Plateau's fascination, when, as a schoolboy, I similarly experimented with wire frames and soap films.

Plateau describes the formation of soap bubbles within the wire frames (*polyèdre laminaire intérieur*, or internal soap film polyhedra), mentioning tetrahedral, cubical and dodecahedral bubbles as examples. He mentions that he has observed altogether 15 different bubble shapes formed on various wire frames (Tome premiere, §§203–204, pp. 361–365 [4]).

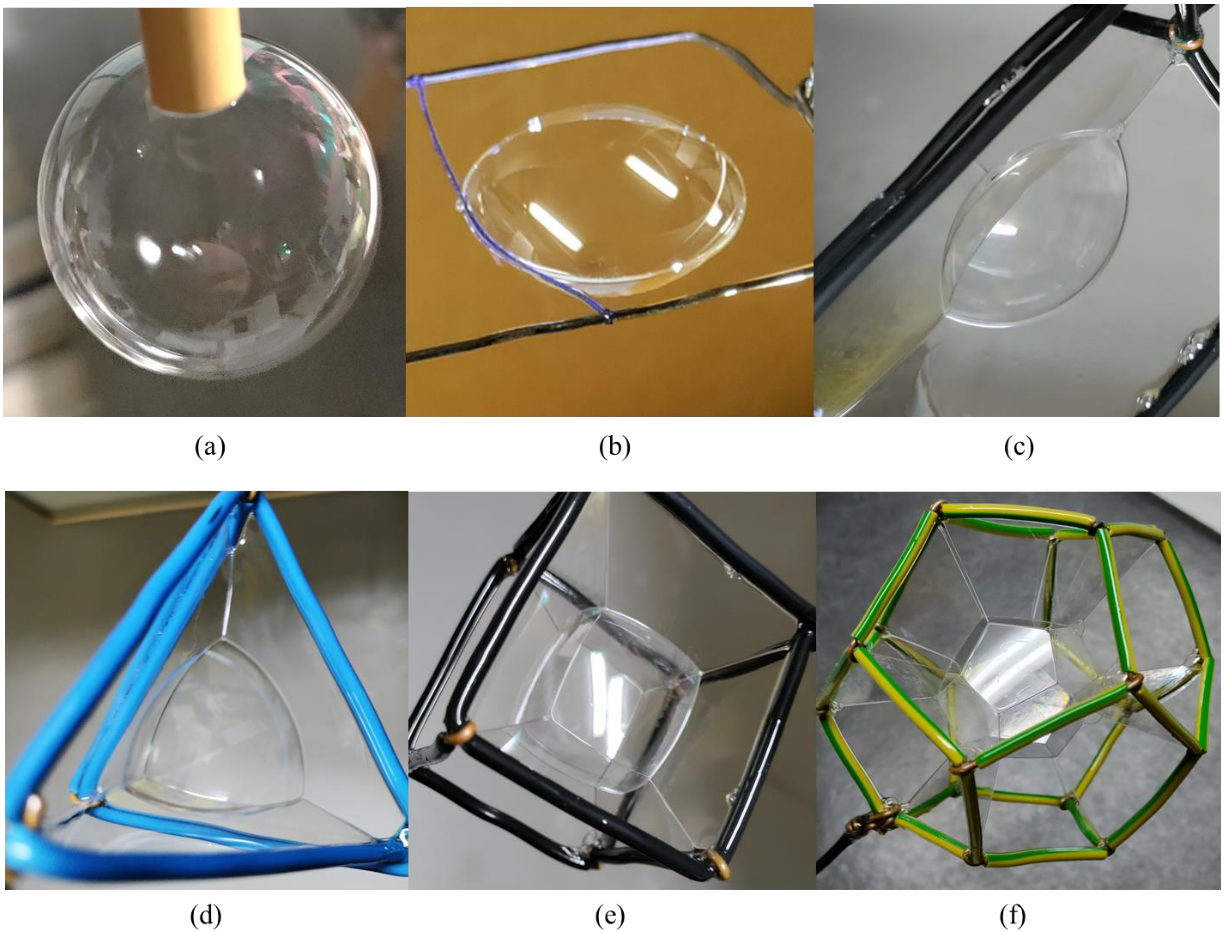


Figure 2: *The six regular bubble shapes created as real soap film bubbles.*

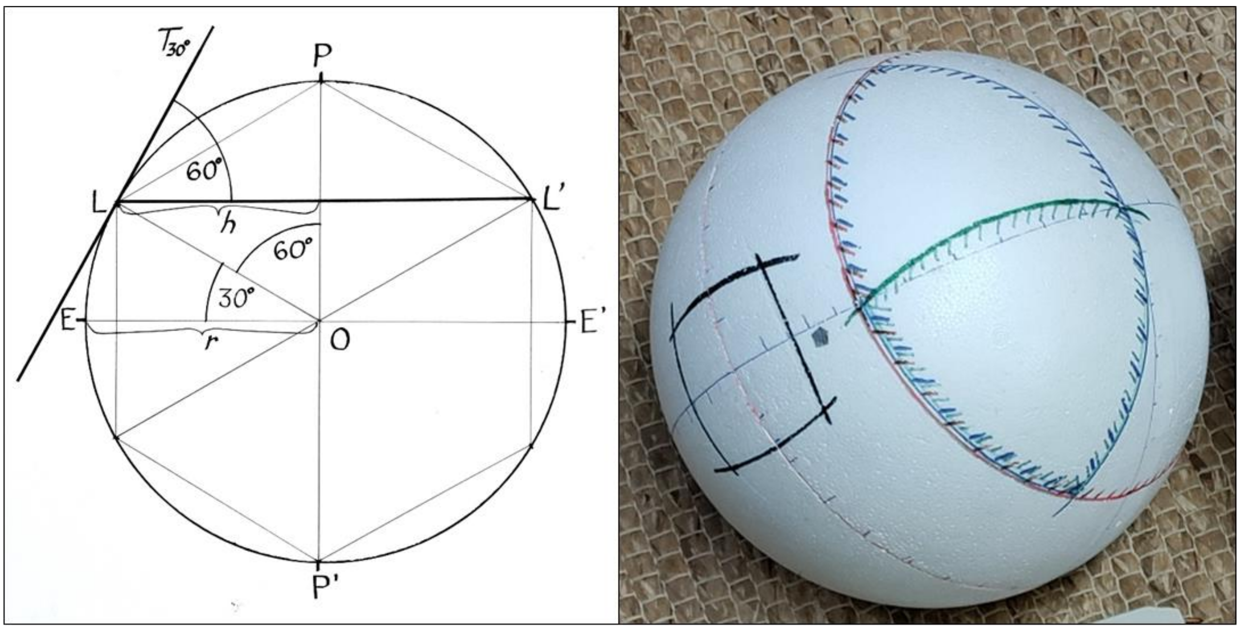
Figure 3: *Wooden models of the regular bubble shapes, sculpted by the author*.

Six Regular Bubbles

The simple sphere included, there are only six regular or isohedral bubble shapes that conform to Plateau's laws. Surprisingly, not Plateau nor any of the later commonly quoted authors (e.g. [1][2][3]) seem to have noticed the existence of this special set of bubble shapes, even though separate mentions of e.g. tetrahedral or cubical bubbles repeatedly occur in the literature. To my mind, this series of regular bubble shapes comprises an interesting parallel to the classical Platonic solids, the five regular polyhedra.

I propose naming the 'Platonic' bubbles with the epithet *-sphere*: *monosphere* (or *sphere*), *di-*, *tri-*, *tetra-*, *hexa-* and *dodecasphere*. These are shown in Figure 2 as soap bubbles on metal wire frames, and in Figure 3 modelled in wood. The wooden models will be shown in the Bridges 2025 Art Exhibition.

The striking harmony of the regular bubble shapes emanates from the invariant features shared and repeating in their geometries. Firstly, all the faces of an isohedral bubble share a constant spherical curvature, or equal radii (r in Figure 4 (a)). Secondly, all vertices are identical throughout the series. And thirdly, each edge is a circle segment with a radius h proportional to that of the sphere: $h = (\sqrt{3}/2) \cdot r$. This is because each edge equals the 30° latitude curve on the bubble face sphere, as shown in Figure 4 (a). Figure 4 (b) shows the geometries of the bubble face elements drawn on a spherical surface.



(a) (b)

Figure 4: *Geometry of the bubble faces. (a) Cross section of a bubble face sphere:* P–P' = *polar axis;* E–E' = *equatorial plane; r = sphere radius;* L–L' = *the 30° latitude plane with radius h;* $T_{30°}$ *= tangent at 30° latitude. (b) Face elements of disphere (red contour), trisphere (blue), tetrasphere (green) and hexasphere (black) drawn on a styrofoam ball. The small pentagonal patch shows the approximate size of a dodecasphere face on this surface.*

The *disphere* consists of two calottes delimited by the 30° latitude. The calotte shape is obtained by drawing with a compass on the sphere surface a circle with a chordal radius equalling the sphere's radius r (cf. Figure 4 (a): P–L = r). Slicing the calotte off from a ball with a cut parallel to the latitude plane results in a 60° angle to its margin. This adds up to the required 120° edge angle when two calottes are joined to form the lens-shaped disphere.

The other edge of the lune- or boat-shaped face of a *trisphere* is drawn by placing the compass needle point on the margin of the disphere calotte, and drawing with the radius *r* an arc segment inside the calotte. Joining three such segments forms a trisphere. As the chordal radius of each edge equals the radius *r* of the face surfaces, the curvature center of each trisphere face is the middle point of the opposite edge.

The face element of the *tetrasphere* is obtained from that of the trisphere by drawing from its one corner an arc with radius *r* across the face. The center point of the curvature of each tetrasphere face is thus the opposite vertex. This shape is commonly called *spherical tetrahedron* or *Reuleaux tetrahedron*.

As regards the *hexasphere*, the cross section of each of its quadrilateral faces equals a 30° sector of the great circle of the sphere (the cross section of the cubical bubble being a bulging square with four 120-degree corners; their supplementary angles add up to 240°, leaving 120°/4 = 30° for each side to complement the full rounding of 360°).

The last one, the 12-sided *dodecasphere* only just fits into Plateau's system; as a regular bubble it comes very close to the shape of the Platonic dodecahedron. In a dodecahedron the dihedral angle is ≈ 116.56°, or only slightly sharper than the bubble-shape's 120°. Likewise, each corner angle of a regular pentagon is 108°, while the bubble faces meet at ≈ 109.5°. The faces of the dodecasphere appear almost planar (cf. Figure 2 (f)) and, as a consequence, very small in size on the sphere surface (Figure 4 (b)). According to my own crude calculations one edge of the pentagonal face equals a central angle of ≈ 2.66° of the great circle of the sphere. Even Plateau himself seems to have skipped calculation of the geodesics of this shape. Plateau succinctly describes the spherical geometric proportions of the tetrahedral and cubical bubbles, but for the dodecahedral one he simply states: *"Dans la charpente du dodécaèdre régulier, le dodécaèdre laminaire intérieur a ses faces de courbure sphérique, mais d'un très-grand rayon*."(the faces of an internal soap film dodecahedron formed within a regular dodecahedral frame have spherical curvature with a very long radius) (Tome premiere §204, 6°, p. 365 [4]).

Platonic Bubbles and Euler's Law for Polyhedra

The three simplest shapes (mono-, di- and trisphere) do not have counterparts among the Platonic solids, whereas of the latter, octahedron and icosahedron do not agree with Plateau's bubble rules, having more than three faces meeting at a vertex.

Euler's universal formula for convex polyhedra, $F + V - E = 2$ (*F*aces plus *V*ertexes minus *E*dges), applies even for the trisphere (3 + 2 – 3 = 2) as well as self-evidently for the tetra-, hexa- and dodecaspheres. For the mono- and disphere bubbles the result is 1 (1 + 0 – 0 and 2 + 0 – 1, respectively), so they do not conform to Euler's law for polyhedra. Lack of vertices may be a key to this deviation.

References

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