The Creation of Ruled Surfaces in a Hall of Mirrors

Ivan De Boi and Rudi Penne

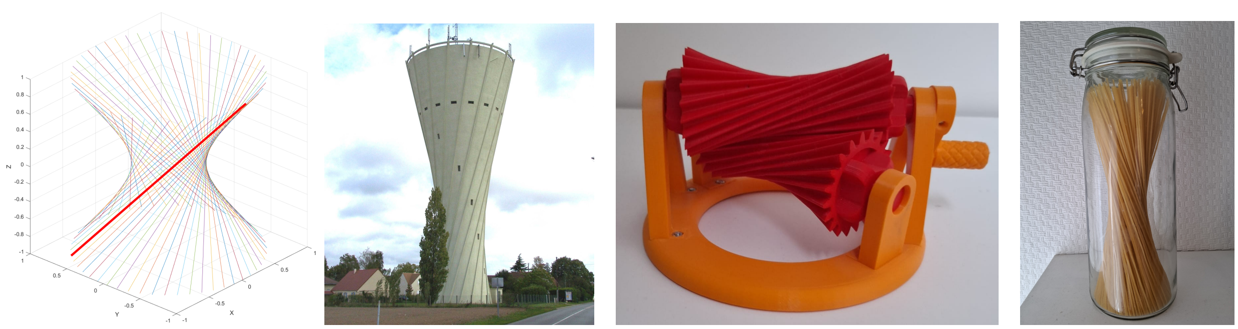
University of Antwerp, Belgium, InViLab; ivan.deboi@uantwerpen.be

Abstract

A two-mirror laser scanner, utilising rotating mirrors to direct a fixed laser, generates a two-degree-of-freedom line set: a *two-mirror congruence*. This congruence, formed by the rulings of coaxial hyperboloids, enables the creation of diverse ruled surfaces. We present a user-friendly application for interactive mirror manipulation, allowing dynamic animation and custom ruled surface generation.

Introduction

Ruled surfaces are surfaces that are swept by a straight line moving in Euclidean 3-space. A classic example is a one-sheeted hyperboloid of revolution (Figure 1a), which is generated by rotating a line that is skew with respect to the axis of rotation. They are the elementary ingredients of this article and will be briefly referred to as "hyperboloids". The rulers of a hyperboloid are said to have 1 degree of freedom, since each ruler is determined by 1 parameter (the rotation angle of the sweeping line).

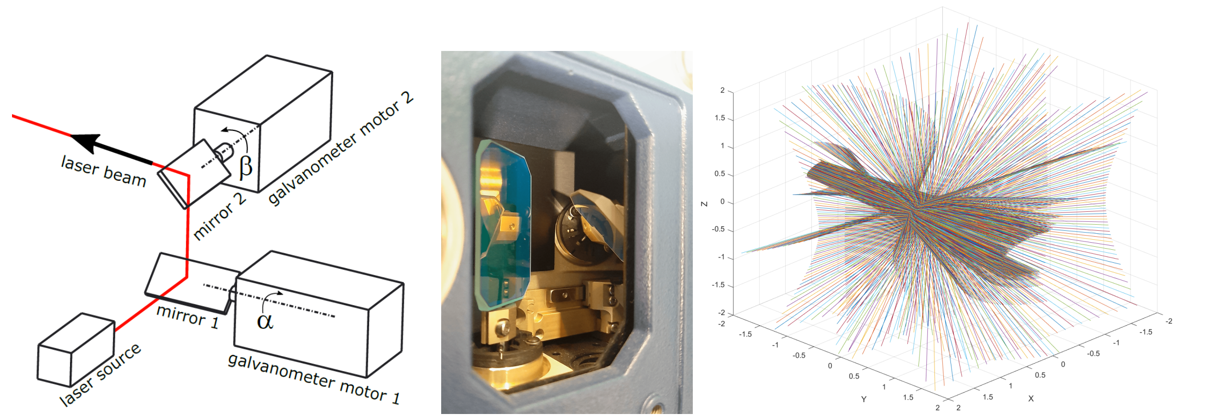


(a) **(b)** **(c)** **(d)**

Figure 1: *(a) A one-sheeted hyperboloid of revolution, swept out by a skew line. (b) Water tower Château d'eau des Essarts-le-Roi (Yvelines, France), image by Henry Salomé. (c) Two 3D printed hyperbolic gears. (d) Spaghetti in a jar.*

This shape has been the inspiration for various artists and architects. A nice treatise on this can be found in the Bridges 2022 contribution by Frank A. Farris [3] and the 2023 contribution by George Hart [4]. We can find hyperboloids in the construction of various towers (Figure 1b), hyperbolic gears (Figure 1c) and even in spaghetti in a jar (Figure 1d). But our interest in hyperboloids is motivated by sensors and other devices that make use of rotating lasers. Often, these laser lines are the reflections of a single fixed laser by one or more rotating mirrors, popular for light effects at concerts or parties.

A *two-mirror laser scanner* controls an incoming laser by two rotating mirrors (around fixed device axes) (Figures 2a and 2b). The set of outgoing laser lines is determined by the geometric setup of the incoming laser and the two mirror axes. In classic geometry [6], such a line variety is called a *line congruence*, because it has 2 degrees of freedom, in the sense that each of its lines is determined by the choices for both mirror angles. In the specific case of doubly reflected laser lines we coin the term *two-mirror-congruence*.



(a) **(b)** **(c)**

Figure 2: *(a) Two mirrors guiding a laser beam. (b) Two mirrors in a Polytec Laser Doppler Vibrometer Scanner PSV400. (c) 10 hyperboloids on a two-mirror-congruence with each 100 lines.*

The visualisation of a two-mirror-congruence is discomforted due to the 2 degrees of freedom, resulting in a sloppy haystack (Figure 2c). On the other hand, it provides an unlimited supply of ruled surfaces, each of which is a subset of the congruence with 1 degree of freedom. We have already mentioned the hyperboloids that are swept by fixing one mirror and rotating the other. But if we rotate both mirrors simultaneously, we create more exotic, sometimes extremely beautiful ruled surfaces, swept by a line that hops between different hyperboloids. In this work we propose a user-friendly application in which you can easily manipulate the two rotating mirrors and observe the resulting outgoing lines.

The Mathematics

If a fixed incoming laser hits a mirror that continuously rotates around a fixed axis (contained in the mirror plane), then it generates a set of reflected laser lines. Observe that the same line variety is obtained by rotating one selected laser line around the given mirror axis, rather than rotating the mirror. This implies that the reflections of one fixed laser by one rotating mirror sweep a ruled surface of revolution, being a hyperboloid in general. In a two-mirror laser scanner, the first rotation angle α determines the first laser reflection, and hence the incoming ray on the second mirror, yielding a ruled surface of revolution when rotating this second mirror. Consequently, in general, a two-mirror-congruence can be partitioned into a family of coaxial hyperboloids, with the common axis of revolution equal to the rotation axis of the second mirror.

In this section, we sketch the procedure to compute subvarieties of this two-mirror-congruence from a user-guided trajectory of angle pairs. For a more profound background and mathematical proofs we refer to our previous work [5].

1. For each mirror, we select and fix three angles of rotation: $\{\alpha_1, \alpha_2, \alpha_3\}$ for the first, and $\{\beta_1, \beta_2, \beta_3\}$ for the second mirror. Each of these triples can be represented by three points on two circles. (A_1, A_2, A_2) on the "α-circle" and (B_1, B_2, B_3) on the "β-circle".
2. The rotation of the two mirrors corresponds to the motion of a point A on the first circle and of a point B on the second circle. The chosen triple bases of angles (or circle points) enable us to express each mirror position as a linear combination,

$$A = x_a A_1 + y_a A_2 + z_a A_3, \qquad (1)$$

$$B = x_b B_1 + y_b B_2 + z_b B_3, \qquad (2)$$

with $x_a + y_a + z_a = x_b + y_b + z_b = 1$. Furthermore, these circle relations apply directly to the corresponding outgoing lasers.

3. The two selected triple bases imply 9 combinations of angle pairs (α_i, β_j), each of which determines an outgoing laser L_{ij}. In a preprocessing phase, we measure or virtually compute these 9 lines and represent them by a 6-vector of (normalised) Plücker coordinates. Now we can generate each line of the two-mirror-congruence as a linear combination of these 9 Plücker vectors.
4. Manipulating both mirrors yields trajectories $A(t)$ and $B(t)$ on both circles. The corresponding (time-varying) coefficients in the relation of Eqn. 2 are copied and applied to the Plücker vectors that represent the basis of the 9 lines L_{ij},

$$\begin{aligned} L_1 &= x_b(t)L_{11} + y_b(t)L_{12} + z_b(t)L_{13}, \\ L_2 &= x_b(t)L_{21} + y_b(t)L_{22} + z_b(t)L_{23}, \\ L_3 &= x_b(t)L_{31} + y_b(t)L_{32} + z_b(t)L_{33}, \end{aligned}$$

generating the outgoing laser L by means of Eqn. 1:

$$L = x_a(t)L_1 + y_a(t)L_2 + z_a(t)L_3.$$

The Application

We built the application in the game engine Unity (version 6.0) and opted for a browser app, which we published on the gaming platform itch.io [1] (will be made public upon publication of this paper). The nine lines needed in the procedure explained above are the result of reflected raycasts calculated by Unity's physics engine. These 3×3 lines and corresponding $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ angles are then used to calculate a new line for any given pair $(\alpha_{new}, \beta_{new})$. We do this for every combination of integer α- and β-values from zero to 360. This yields a 360×360 set of lines.

We allow the user to control the angular velocities ω_α and ω_β of the two rotating mirrors by means of two sliders. Setting them both to zero means observing a fixed set of mirror angles α and β thus a fixed straight line on the two-mirror-congruence. Setting one to a non-zero value means the resulting line rotates around its axis at the given angular velocity. Making both angular velocities non-zero results in a line that travels on the two-mirror-congruence. This means continuously changing hyperboloids at a rate ω_α and travelling on those hyperboloids at a rate ω_β. This allows a user to come up with all kinds of interesting traces of a straight line through 3D space, and a very rich generation of animated patterns. The result is always a ruled surface, as we only work with straight lines in 3D space. Some interesting examples are given in Figure 3.

Summary and Conclusions

We presented an application to visualise the set of straight lines that lie on a two-mirror-congruence. We hope this will serve as inspiration to artists creating DJ background walls, designing laser shows, or producing VJ loops, as well as those making abstract expositions with straight lines. We invite everyone to send us their creations by posting them online (YouTube, Instagram,...) and emailing us the link. Please make sure to mention both the Bridges 2025 conference and this article.

Future work includes the manipulation of the mirror rotation axis, which means altering the two-mirror-congruence itself. Another interesting approach would be the manipulation of line elements instead of lines [2]. These consist of both a line and a point on them and thus describe point clouds with normal vectors.

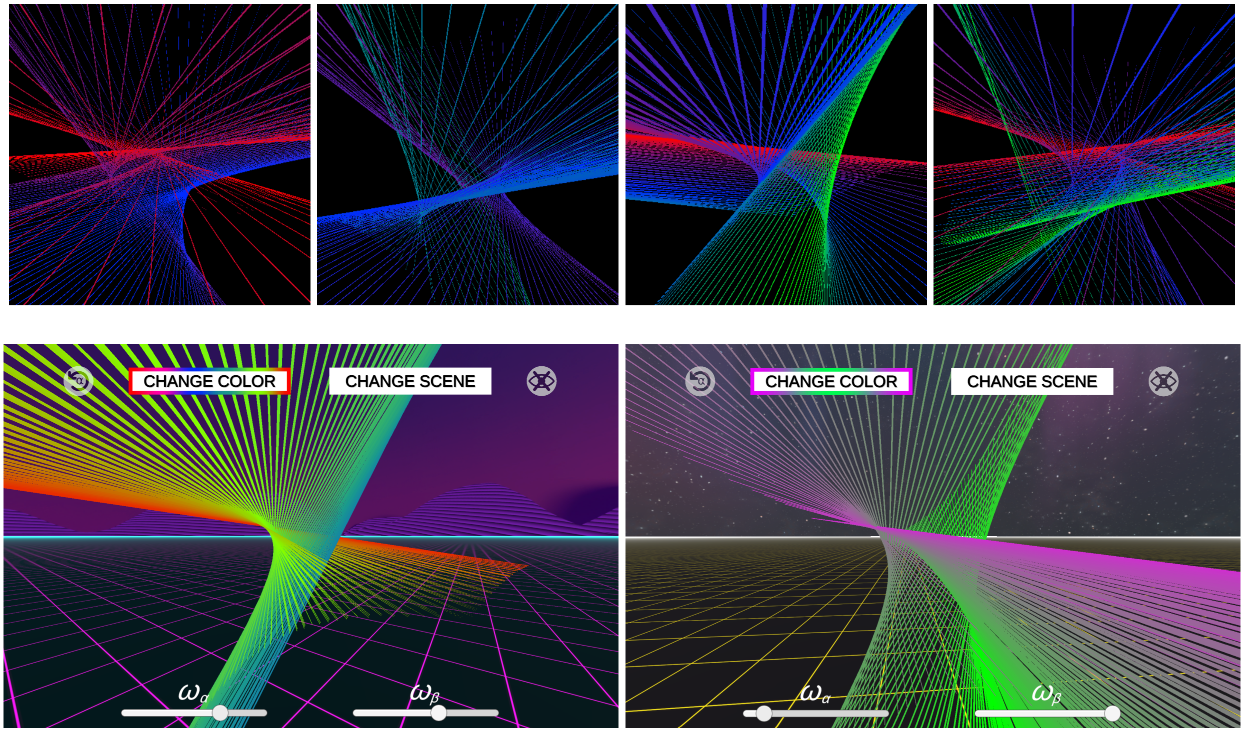


Figure 3: *Some examples of ruled surfaces produced by our app.*

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