# Beyond the Box: Cardboard Math-Art

George Hart[1] and Elisabeth Heathfield[2]

[1]Wiarton, ON, Canada; george@georgehart.com
[2]Wiarton, ON, Canada; elisabetheathfield@gmail.com

## Abstract

This paper shows some of our experiments over the past few years involving ways to use cardboard for math-art projects, stretching the limits of the material and opening people's minds to its creative and artistic potential.

## Introduction and Background

We have been exploring the potential of cardboard for making collaborative, scalable math-art pieces with significant visual impact. We present here a variety of geometric constructions that can be replicated in classrooms, museums, and public spaces for their educational and aesthetic value. With cardboard one can design large-scale structures that are safe for groups to assemble. It is widely available, low-cost, ecologically friendly, and easily cut and assembled. The specific projects below are intended to introduce construction techniques and to inspire people to go further in developing their own ideas.

Our original impetus for using cardboard came from Reza Sarhangi, who in preparing for Bridges 2012 asked one of the authors (Hart) if he could design a group activity for participants to do during a break period. The resulting dodecahedral form assembled from 30 slotted and folded large rectangles is documented in [1] along with some other early designs. Since then, we led a large cardboard Rhombic Triacontahedron workshop at Bridges 2016 in Jyväskylä [3] and we showed a giant cardboard "Wiggly Dome" in our 2017 Bridges paper [4]. Our Making Math Visible web site [2] includes templates for two additional large cardboard constructions, with photos of them being assembled at the 2017 National Math Festival in Washington, DC and at the Fields Institute in Toronto. This paper presents a survey of newer designs. The supplement to this paper provides their cutting templates and some assembly tips.

**Figure 1:** *Cardboard "People" sculpture, 2 meter diameter.*

## Examples

Figure 1 shows a giant (2 meter) orb consisting of 60 tab-and-slot-connected laser-cut cardboard cartoon humanoids. It is based on mathematical foundations of symmetry and polyhedral stellation.

In Figures 2 and 3, workshop participants assemble the five Platonic solids from laser-cut cardboard struts that they fold lengthwise and clamp together before gluing. There is one cardboard piece per edge, rather than one per face as in most paper polyhedron models, giving students a different understanding of the Platonic solid's skeletal structures. Once it is assembled and checked for accuracy, the clamps are removed one at a time, glue is brushed in, and the clamps are replaced for 15 minutes while the glue dries. This activity is a stepping stone towards more advanced polyhedral and geometric constructions.

Figure 4 shows a 10-fold polar zonohedron dome composed of laser-cut cardboard rhombi. It is temporarily held together with binder clamps before reassembly as a classroom activity.

Figure 5 is an open-face ("Leonardo style") model showing the structure of the 120-cell, one of the six regular convex polytopes in 4-dimensional space. This sculptural presentation lucidly presents 16 of its cells, which is sufficient to show the five types of projected cells and the various ways they connect. The laser-cut face panels are simply butted edge-to-edge and tacked together with hot-melt glue.

**Figure 2:** *Assembling polyhedra edge models.*

**Figure 3:** *Octahedron edge model.*

**Figure 4:** *Zonohedron dome, 2 meter diam.*

**Figure 5:** *Sixteen connected cells of the 120-cell.*

**Figure 6:** *Cardboard icosidodecahedron.*

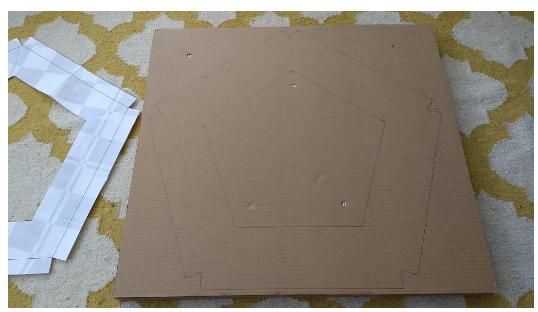

**Figure 7:** *Paper template and stack of cardboard to be cut to make pentagon pieces.*

Figure 6 shows a large icosidodecahedron assembled from 20 triangles and 12 hollow pentagons. Archimedean solids are easy to assemble like this with binder clamps on the inside, then later disassembled for storage or glued for permanent display. Figure 7 indicates how to cut a stack of large cardboard pieces all at once. Note the screw heads. A paper template was traced on to the top sheet of a stack that was screwed together with sheet-rock screws and cut through on a bandsaw. The inner pentagon was cut with a hand jig-saw, then the individual flap lines were scored partway through and folded.

**Figure 8:** *Catenary Arch.*

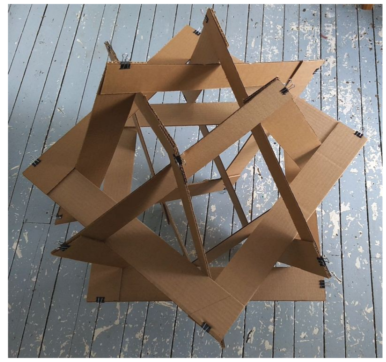

**Figure 9:** *Regular Polylink.*

**Figure 10:** *Tree.*

**Figure 11:** *Untitled cardboard sculpture.*

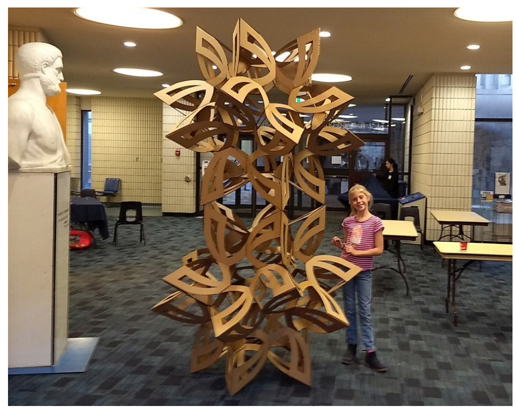

**Figure 12:** *Two mirror images of "Autumn".*

Figure 8 shows a catenary arch assembled by teachers at a Math for America workshop in NY City. We developed this after three earlier variations that we presented at Bridges 2018 in Stockholm [5]. This model is larger and joined with packing tape. It comprises 39 laser-cut hollow quadrilaterals assembled into triangular modules and joined to make an elegant lightweight structure.

An interesting puzzle one can make using cardboard is the regular polylink of six hollow squares shown in Figure 9. Each square is assembled from four long pieces, making the total inventory of parts just 24 identical cardboard rectangles. Although the parts are trivial to make, the puzzle is quite a challenge to put together. We have written up a version of this puzzle workshop that uses craft sticks on our Making Math Visible page [2], but enlarging it to be several feet across turns it into a fun cooperation challenge for a small group of puzzle lovers. It is shown here with clamps, before gluing.

Figure 10 shows how cardboard can be used to make a simple pagan holiday tree. Each of its 21 stacked layers is a pentagram made from five slotted rectangles. The design is a nice exercise in similarity and geometric scaling, with each layer reduced 10% from the next larger layer. It is is a lovely Zeno-like way to illustrate that no matter how many layers you add, you will never reach the apex.

Figures 11 and 12 show how cardboard can be used to make impressive gallery-worthy sculpture. In each, there are two congruent mirror-image versions of the same design. The sculpture in Figure 13 was built by students at St. Andrew's School in Delaware, who first cut out the 120 cardboard components using scroll saws. Figure 12 shows two stacked copies of a sculpture we call Autumn [2] that was assembled from laser-cut pieces at a public event at the University of Toronto. In both cases, because of their light weight, we could easily install them as a temporary pop-up just by tying them to the ceiling.

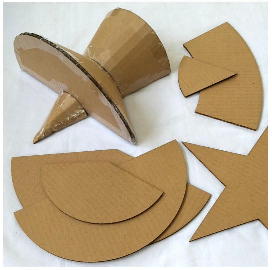

**Fig 13:** *Streptohedron*

**Figure 14:** *Orb.*

**Figure 15:** *Curved gluing jig for Orb.*

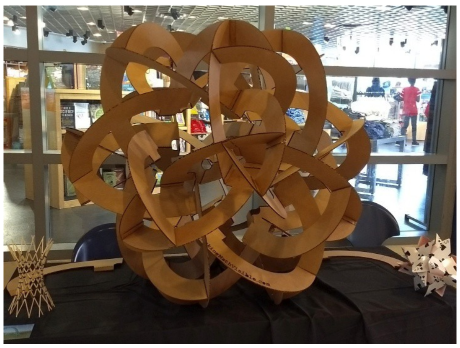

**Figure 16:** *World Maker Faire, 2017.*

**Figure 17:** *National Math Festival, 2019.*

In Figure 13, we show a very different sort of mathematical object: half a "streptohedron", a solid of revolution generated in this case by a pentagram. Its conically curved surfaces make cardboard an ideal material for this form. Calculating the lengths and angles for these cones is an excellent exercise in applied geometry at the high school level.

Figure 14 shows another structure using curved cardboard. With very little material, one can create an enormous orb that illustrates a classic non-edge-to-edge tiling of the sphere. The 12 pentagons and 20 triangles have different edge lengths, because the bent strips meet at their 1/3 and 2/3 points. Figure 15 shows how each of the 30 units was laminated from three long strips that were glued, sandwiched together, and clamped in a curved jig while the glue dried.

Figures 16 and 17 show sculptures designed for major public events. Figure 16 was built by visitors to the 2017 World Maker Faire in New York City. Figure 17 was made at the 2019 National Math Festival in Washington, DC. Both illustrate how cardboard can create elegant math-art objects.

## Conclusions

Cardboard can be seen in some ways as a metaphor for traditional math education. People might view it as boring, rectangular, flat, and utilitarian, but it can be transformed if approached creatively. Just as cardboard can make more than simple boxes, so math education can be so much more than what is taught in the conventional classroom. One future project we are thinking about is a pop-up "math café" filled with mathematical structures that we would document so others could replicate them. We hope viewers will see beyond the box to find the potential for beauty in this common material.

## References

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