Collaboration on a Mathematical Sarasvati Allegory

Amrita Acharyya[1] and Uli Gaenshirt[2]

[1] Department of Mathematics and Statistics, University of Toledo, Ohio 43606, USA;
amrita.acharyya@utoledo.edu
[2]Sculptor & Researcher, Nuremberg, Germany; uli.gaenshirt@gmail.com

Abstract

We describe and analyze a picture, created in collaboration and mutually inspired by the images we presented at the Bridges Conference 2024 in Richmond. The geometry of the original hand-drawing was recomposed and supplemented with suitable fractal objects, further images and ornaments. All parts of the pictorial program are linked by mathematical aspects. The result offers the viewer a comprehensive picture of geometric facts by means of artistic representation, where the authors were guided by traditional allegorical figures from various cultures.

Introduction

Examples of mathematical illustrations are rare in antiquity and the Middle Ages. They reached their first peak in the Renaissance in the form of allegorical figures, illustrations of mathematical concepts and perspective depictions of geometric devices. In our times, the mathematical knowledge has developped enormously. The digital possibilities allow us to combine hand drawings, photos and computer-generated graphics without great difficulty. Data exchange enables direct artistic collaboration across all borders.

We present and describe here an image [1] that can be understood as a synthesis of the images from both of us [2][4]. But it is also a reflection on the mathematical and artistic exchange that we practiced digitally in the period after the Bridges Conference in Richmond in 2024.

Figure 1: *The mathematical Sarasvati allegory. The correlations of the three r- and four x-coordinate systems are shown seperately in the margins. The actresses indicate the crucial points of interaction.*

Initiation, Inspiration and the Development of the Picture Idea

The initial spark for this image was a hand-drawn birthday present that one of us (Amrita) drew for the other one (Uli) the day after the 2024 Bridges conference in Richmond VA. Inspired by the cardioid that outlines the portrait, the idea was born to surround the face with an aura from the fractal Mandelbrot set [3][5], with the same cardioid inscribed. The ladies crowning the portrait got math problems to solve. Sarasvati, who is among many other things a professor of mathematics and arts, explains the geometry. Finally the whole scene was moved above the cloud cover over the city of Richmond.

The Mathematical Objects and their Artistic Interaction with the Allegorical Figures

All mathematical objects in this picture are related to each other, be they curves, axes, lines or fractals. The fractal Mandelbrot set is represented in three different sizes and four different contexts (see Figure 1). We provide formulas for all objects and carry out calculations using comprehensible numerical examples. We also describe the rotations and compressions that are neccessary to leave the hand-drawn elements unchanged while connecting the mathematical objects and the female actors in a meaningful way.

The Equation of the Cardioid in the Cartesian Coordinate System and in the Complex Plane

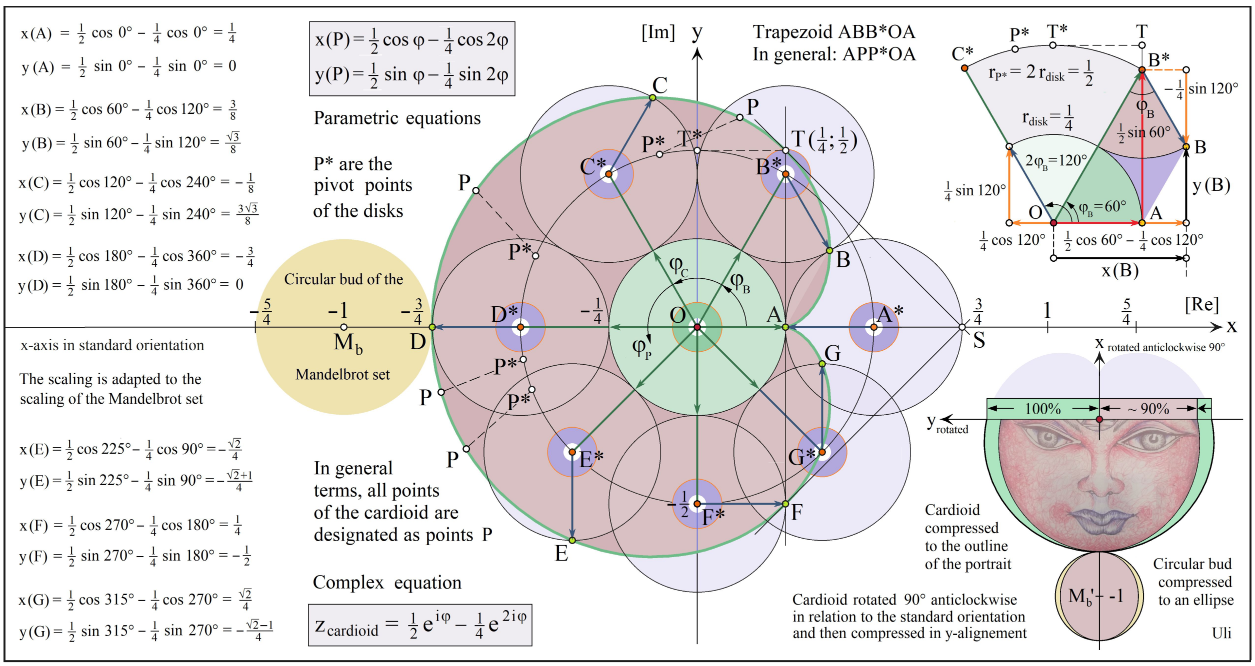

Figure 2: *The points A, B, C, D, E, F, G (in general: P) of the cardioid with the corresponding positions of the seven circumferential disks. Their x and y coordinates are calculated by parametric equations.*

The green outlined, lila filled cardioid in Figure 2 is created by rolling a circular disk (grey or transparent) on a fixed disk of the same size (green). We illustrate the construction here in detail, as the coordinate systems in the usual presentations mostly do not correspond to that of the complex Mandelbrot set.

The radial arrows of the two disks with the centers in O and A^* meet on the x-axis at the point A. When rolling around the green disk, the arrows of the six other exemplary disk positions point to the dots B, C, D, E, F and G, i.e. the points P at the arrowheads of all possible disk positions form the cardioid. After the rolling disk has completely circled the fixed disk, the arrowheads of both disks meet again at point A. At the same time, the rolling disk has finished a double rotation around itself. Figure 2, top right, illustrates the calculation of the x and y values of point B by the equations at the top left, with $\varphi_B = 60°$. Figure 2, below right, shows the 90 degree counterclockwise rotation of the cardioid in relation to the standard orientation and the y-compression required to coincide with the hand-drawn portrait.

The Developement of the Allegorical Figures in the Context of an Artistic Composition

The central portrait that carries the two ladies with their crossing braids as a headdress was the initial composition of the picture, while the portrait fitted into the fractal Mandelbrot set was the birth of the conceptual idea. The red bindi of the portrait image is at the origin of the coordinate system of the added Mandelbrot set, which is rotated 90 degrees counterclockwise compared to the standard orientation.

Point *S* in Figure 2, that marks the intersection of the tangents $\overline{TS}$ and $\overline{FS}$ of the cardioid, is the point at which the ladies' braids cross. The redesigned headdress figures were composed along these diagonals. Their outstretched arms build a bridge between the upper Sarasvati figures, which were added last. The hand gestures and the repeated appearance of Sarasvati identify her as an omnipresent teacher and give the two ladies in their blue trouser suits the character of inquisitive and intelligent students.

The vina, Sarasvati's traditional string instrument, corresponds with the vertical *r*- and *x*-axes, whose correlations with one another are shown separately at the side edges of Figure 1. Another example of such an artistically motivated formal interaction are the increasing sizes of the young swans at the lower lateral edges of the picture, which are inspired by the different sizes of the Mandelbrot buds.

Geometric Insights of the two Headdress Ladies into the Secrets of the Mandelbrot Set

The lady on the left holds with her left hand the green Julia fractal, which is defined in the complex plane $\mathbb{C}$ by the recursive equation $z_{n+1} = z_n^2 + c$, with the complex constant $c = (0; i)$ [5]. Near this point, the shape of the green Julia set is similar to the shape of the dark violet, small-sized Mandelbrot set. Both sets are shown in the same coordinate system in their standard orientation.

The green Julia set depends from that start-values z_0 for which z_n shows a clearly defined behavior. The Mandelbrot set is based on the same short equation $z_{n+1} = z_n^2 + c$, but consists of all complex constants c for which any complex number z_n (with lim $n \to \infty$) is bounded by the circle with radius 2. If we always start with $z_0 = 0$, this is the case for the real numbers 0, -1, -2, but not for 1! This can be shown as follows: $z_0(1) = 0,\ z_1(1) = 1,\ z_2(1) = 2,\ z_3(1) = 5,\ z_4(1) = 26,$ i.e.: $z_3 > 2,\ z_4 > 2, \ldots,\ z_{n+3} > 2,$ with $n \in \mathbb{N}_0$.

Figure 3: *(a) Small-sized Mandelbrot set in standard orientation with overlaid green colored Julia set. (b) Mid-sized Mandelbrot set clockwise rotated by 90 degrees with the correlated bifurcation tree.*

The lady on the right side examines the correspondance between the Mandelbrot set and the bifurcation diagram of the logistic equation, that reads: $x_{n+1} = rx_n\,(1\text{-}x_n)$. This equation can be used to describe chaotic demographic developments. The values of the *r*-axis of the logistic equation correlate with the values of the opposite *x*-axis of the medium-sized Mandelbrot set. The strip near $r = 3.82$, which runs horizontally through the black chaotic values of the logistic equation, is a window of periodic order. Its counterpart is a mini Mandelbrot set that belongs to the medium-sized Mandelbrot set and is located where the line of the *r*-value 3,82 intersects the *x*-axis of the Mandelbrot set at the downward-pointing arrowhead.

In the image of the Mandelbrot set (png, 8192 x 8192) by Aubin Arroyo [3], the mini- and the two micro-Mandelbrot sets are clearly visible; they correlate exactly to the windows of the bifurcation tree! This correlation is more than just surprising. Aubin's following pictures of details in his gallery show that mathematics itself is a highly talented artist, in line with our personification by the central portrait.

Another interesting, interactive website of a Suiss school [5] describes the relationship of the Mandelbrot set and the Julia sets. We provide some basic information in English here: After clicking on the picture of a Julia set, the Mandelbrot set appears. You can then click there to create the Julia set that corresponds to the coordinates of the click. You can get interesting Julia sets by clicking on the edge of the Mandelbrot set. Please click on “Juliamenge” if you have changed the values of Re(c) or Im(c).

The Image Background

On the one hand, the most interesting worlds of thought are detached from earthly burdens; on the other, the presented image is inseparabably linked to Richmond for us. We therefore thought it appropriate to superimpose the scenery over the cloud cover of Richmond as seen from the airplane. The cloud structure is reflected on the axes of the coordinate systems, which lends the image a surreal drama.

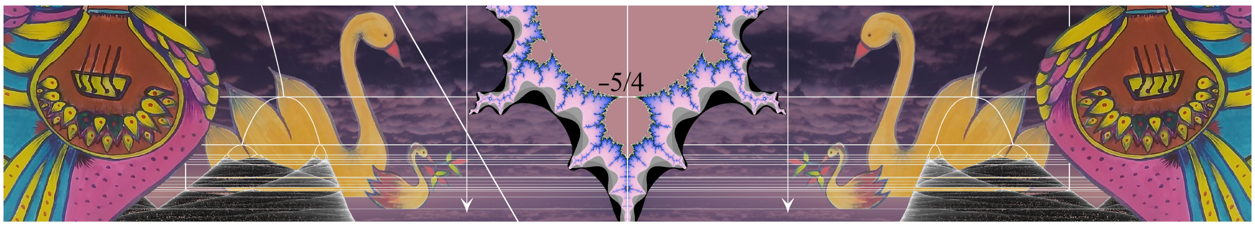

Figure 4: *The correspondence of the cloud ceilings with the bifurcation diagram.*

The horizontal lines at the bottom of the image suggest that the cloud formations behind them correlate with both the Mandelbrot set and the upside down bifurcation trees, but the coincidence is based on the perspective of the photo. In this respect, the interplay is only the result of a conscious artistic design and not a genuine correlation, such as exists between the Mandelbrot set and the logistic equation.

Conclusion and Outlook

In this paper, we tried to find out what possibilities lie in mathematical illustration if we give priority to the artistic aspect and use the mathematical aspect to connect the different ideas. But we do not want this to be understood in any way as a guideline. We plead here for the absolute freedom of the arts and the use of their enormous bandwidth for the artistic development of mathematical art and we suggest cooperation wherever this seems possible.

Acknowledgements

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