Artistic Nets to Create Ribboned Regular Tetrahedrons

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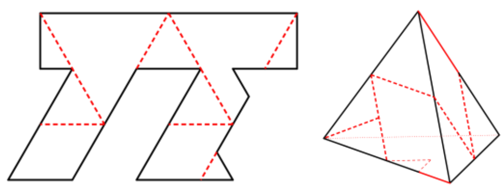
Abstract

In this paper, we propose an unexpected way to make a regular tetrahedron by just folding a long and narrow strip of paper. We show how to calibrate the dimensions of the strip to produce different folding patterns. Furthermore, if we carefully cut the paper strip into two pieces, then two ribboned regular tetrahedrons will be generated.

Relationship between Nets of a Regular Tetrahedron and Tilings

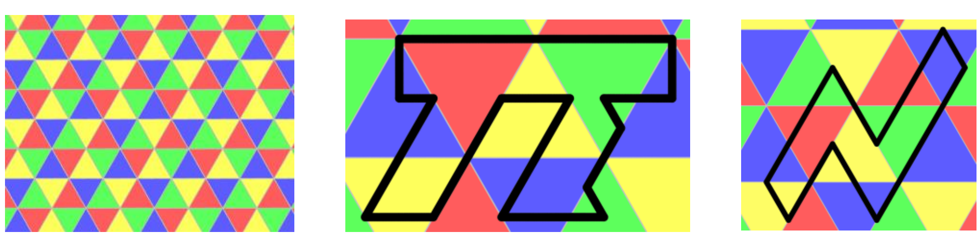
In general, there are only two types of nets of a regular tetrahedron, which can be folded (along edges) to become the faces of the regular tetrahedron. But, if we allow cuts across the faces, then we can have lots of artistic nets. For example, a net shaped like the Greek letter π is shown in Figure 1.



(*a*) (*b*)

Figure 1: *An artistic net of a regular tetrahedron*:
(*a*) *the* π*-net*, (*b*) *folding up the* π*-net*

In [3], we obtain a way to create artistic nets of a regular tetrahedron. The *hexagram floor* is a 4-coloring of a tessellation of equilateral triangles, such that every six equilateral triangles around a regular hexagon (of six equilateral triangles) all have the same color, as shown in Figure 2(*a*). If a diagram, whose area is equal to the sum of areas of four equilateral triangles, can be colored as the hexagram floor such that all parts of the same color inside it can form an equilateral triangle by specific translations and rotations, then this diagram is a net of a regular tetrahedron. Figures 2(*b*) and 2(*c*) show two examples.



(*a*) (*b*) (*c*)

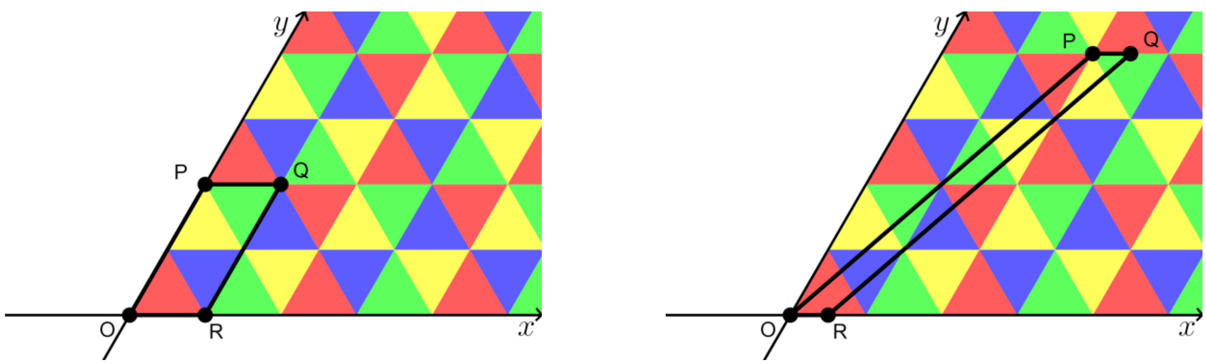
Figure 2: *Some colored nets of a regular tetrahedron found by using the hexagram floor*:
(*a*) *the hexagram floor,* (*b*) π*-net,* (*c*) *N-net*

Colored Parallelogram *OPQR*(*x*, *y*)

Now, we construct an *oblique Cartesian coordinate system* in the plane such that the angle between the x-axis and the y-axis in the first quadrant is 60°. Then we color the first quadrant by using the hexagram floor, where each unit length is equal to the side length of an equilateral triangle. Let O be the origin (0, 0), and P be the point (x, y) where $x \geq 0$ and $y > 0$ are integers. A colored parallelogram, denoted by $OPQR(x, y)$, is defined as follows:

1. O, P, Q, R are the four vertices of $OPQR(x, y)$ (in order) such that the line segment joining P and Q is parallel to the x-axis.
2. The area of $OPQR(x, y)$ is equal to the sum of the areas of four equilateral triangles (of different colors). And the width of the colored parallelogram (or $|\overline{OR}|$) is $2/y$.

For example, Figure 3(*a*) shows $OPQR(0, 2)$, which is one of the traditional two nets of a regular tetrahedron, and Figure 3(*b*) shows $OPQR(2, 4)$.



(*a*) (*b*)

Figure 3: *Colored parallelograms*: (*a*) *OPQR*(0, 2), (*b*) *OPQR*(2, 4)

An *OPQR*(*x*, *y*)-Net of a Regular Tetrahedron

We define an *OPQR*(*x*, *y*)-*net* of a regular tetrahedron as a colored parallelogram $OPQR(x, y)$ which can be folded into a regular tetrahedron. In [4], it has been proved that if the greatest common divisor of $x/2$ and $y/2$ is 1 (i.e., relatively prime), then an $OPQR(x, y)$-net of a regular tetrahedron exists. Otherwise, if $x/2$ and $y/2$ are not relatively prime, then the object folded from $OPQR(x, y)$ will cover some portion of the surface of a regular tetrahedron multiple times, and thus some other portion remains uncovered. For example, folding the strip shown in Figure 4 will double-cover a portion of the surface of the regular tetrahedron.

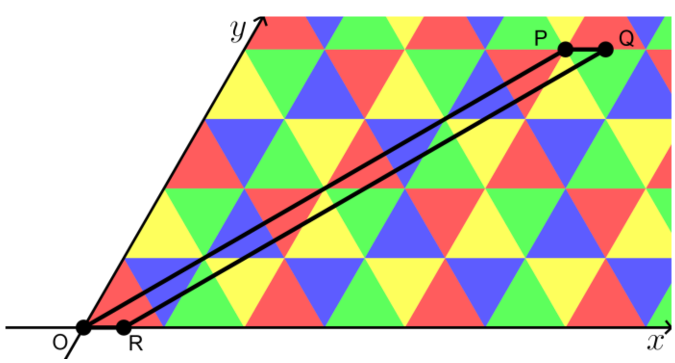
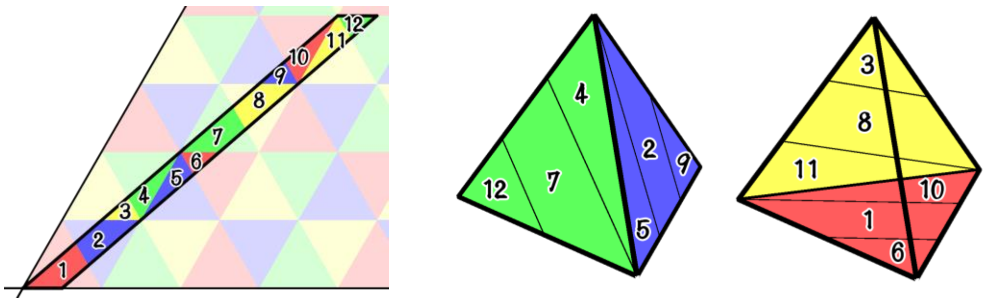


Figure 4: *OPQR*(4, 4)

Folding a Paper Strip into a (Ribboned) Regular Tetrahedron

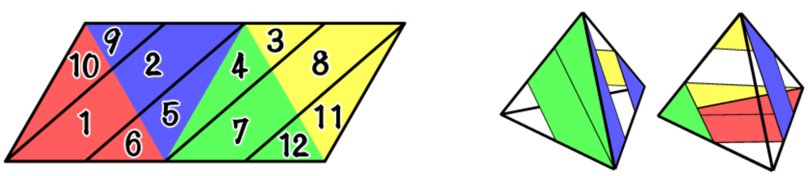
First, we take a paper strip colored as an *OPQR*(x, y)-net of a regular tetrahedron. Next, from the bottom to the top, we number different colored parts of the paper strip in order. Finally, according to the order of numbers, we can fold this paper strip into a regular tetrahedron. Interested readers can download the file from the website in [5], and print it out, then cut the strip along the black lines as in Figure 5(*a*), and fold the paper strip along the color boundaries. A GeoGebra animation has been made for the process, see [6].



(*a*) (*b*)

Figure 5: *Fold a paper strip into a regular tetrahedron*:
(*a*) *numbering a paper strip colored as OPQR*(2, 4)*-net*, (*b*) *folding up*

Since the numbered paper strip will be very long and narrow when x and y are large, we give a rearrangement for a clear understanding by using a traditional net of a regular tetrahedron, as shown in Figure 6(*a*), refer to [3] for details. Moreover, we can also cut the numbered paper strip into two pieces of the same width as the original strip such that each piece forms a ribboned regular tetrahedron. For example, in Figure 6(*b*), we fold parts 4–9 of the numbered paper strip into one ribboned regular tetrahedron, and parts 10–12 and 1–3 into the other one, where parts 4–9 and parts 10–12 + 1–3 are consecutive.



(*a*) (*b*)

Figure 6: *Two ribboned regular tetrahedrons*:
(*a*) *a rearrangement of the numbered parts*, (*b*) *folding parts 4–9 / 10–12 + 1–3 of the paper strip*

For more examples, a rearrangement of a numbered paper strip colored as *OPQR*(2, 20)-net of a regular tetrahedron is given as follows:

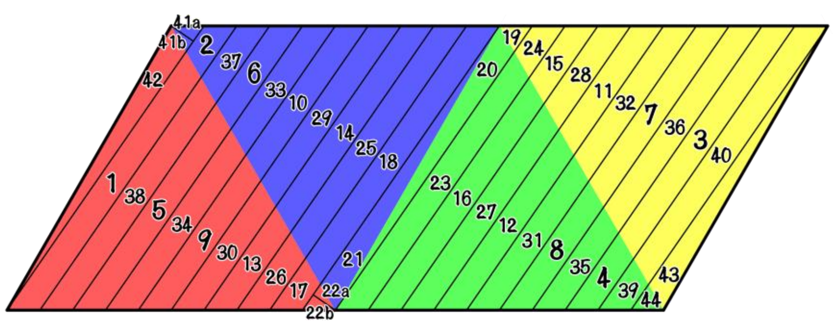
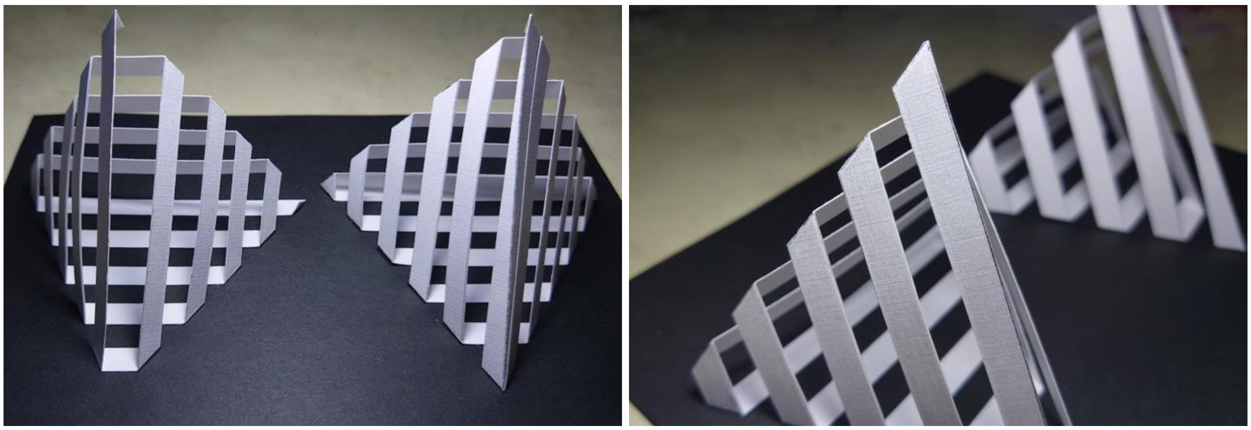


Figure 7: *a rearrangement of a numbered paper strip colored as OPQR*(2, 20)*-net*

In order to enhance the beauty, we cut parts 22 and 41 along line segments perpendicular to the sides of the strip to make two ribboned regular tetrahedrons which are complementary to each other, i.e., they can be combined to cover the regular tetrahedron fully without overlapping. Parts 22b, 23–40, 41a of the strip can be folded into one ribboned regular tetrahedron α, and parts 41b, 42–44, 1–22a can be folded into the other ribboned regular tetrahedron β, see Figure 8. (Similarly, these parts are consecutive.)



(*a*) (*b*)

Figure 8: *Two ribboned regular tetrahedrons*: (*a*) *left: α and right: β*, (*b*) *left: α and right: β*

Summary and Conclusions

We propose a way to fold a paper strip into a regular tetrahedron. First, we construct an oblique Cartesian coordinate system in the plane and color the first quadrant of it by using the hexagram floor. Next, we define an *OPQR*(*x*, *y*)-net of a regular tetrahedron as a colored parallelogram *OPQR*(*x*, *y*) in the first quadrant which can be folded into a regular tetrahedron. Finally, by cutting a numbered paper strip colored as an *OPQR*(*x*, *y*)-net of a regular tetrahedron into two pieces, then we can fold each piece to obtain a ribboned regular tetrahedron respectively.

Acknowledgements

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[6] https://www.geogebra.org/m/zuvjc35f