A Futurama Dance

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Abstract

Dance is a subject that allows for mathematical analysis on many levels. In the following, we'll construct a dance in the English Country/contra dance style that illustrates an elegant mathematical theorem that first appeared on the animated series "Futurama".

The Futurama Theorem

Imagine a sequence of objects in some order, say the letters $abcdef$. Switching any two letters (which need not be adjacent) is called a transposition, which we can record by listing the letters being switched. For example,

$$a\ b\ c\ d\ e\ f \xrightarrow{(ae)} e\ b\ c\ d\ a\ f$$

By applying multiple transpositions, one after the other, we can rearrange (*permute*) the letters and put them in any order; and once the letters have been rearranged, we can try to return to them to their original positions.

Generally speaking, restoring the original positions will require re-using some of the transpositions. But mathematics and art exist on the border between constraint and freedom. So what if we impose the constraint that no transposition be re-used?

This problem appears in "The Prisoner of Benda," an episode of the animated TV series *Futurama* that aired in 2010 [5]. In the episode, a mind-switching device allowed two people to swap minds. However, once a pair of *bodies* swapped the minds they housed, that pair could never again swap contents. Ken Keeler, who co-wrote the episode, holds a Ph.D. in applied mathematics from Harvard [2]. Keeler proved (on network television, no less!) that as long as there are at least two objects that have not participated in any transposition, it is always possible to return all objects (including the "untouched" objects) to their starting point without re-using a transposition.

The fact that order can be restored with the addition of *two* more objects suggests we can create a dance to illustrate Keeler's Futurama Theorem. In the following, we'll go through the steps of choreographing an illustration of the Futurama Theorem in the style of an English Country dance.

Dance

English Country (EC) dance, and its modern descendents, contra dance and square dance, are typically done in sets of couples [1, 4, 6, 7]. Traditionally, these couples consist of a man and a woman, but we'll use the more inclusive (and mnemonic) terminology of "Larks," the partner on the left and "Robins," the partner on the right. There is a first couple $L1, R1$; a second couple $L2, R2$; and so on.

A dance is typically taught (*called*) by specifying an initial spatial configuration: for example, "any number of couples in a line" or "sets of two couples facing each other." The caller then specifies how the dancers move (a *figure*): for example, "First Lark and Second Robin switch places." In the following, we will always refer to dancers by their original location: "First Lark" remains "First Lark" and has partner "First Robin" regardless their current positions.

Most dances end with all dancers back in their original position in the original spatial configuration, allowing dance to be analyzed using spatial symmetries [4, 7]; this type of analysis led to the creation of an entirely new style of dance [6]. For the following, we'll stay within an existing style but choreograph a dance with specific constraints dictated by the Futurama Theorem.

In principle, a dance can be formed from a completely arbitrary sequence of transpositions. However, this would be both difficult to call and difficult to learn. Consequently certain transpositions are favored in EC dance and its descendants. We will focus on the following:

- Two members of a couple may switch places, $(Ln\ Rn)$. We'll refer to this as a "partner switch."
- A figure may switch two Larks, as long as it also switches their partner Robins: $(Ln\ Ln')$ must be accompanied by $(Rn\ Rn')$. We'll refer to this as a "same-bird switch" (referencing the couple numbers n, n' as necessary).
- A figure may switch a Lark and a Robin, as long as it also switches their partners: $(Ln\ Rn')$ must be accompanied by $(Rn\ Ln')$. We'll refer to this as an "opposite-bird switch" (referencing the couple numbers n, n' as necessary).

To illustrate the Futurama theorem, we'll create an EC dance with two parts. In the first part, the dancers move from their original positions in some fashion (but to meet the conditions of the Futurama Theorem, two dancers do not move). In the second part, all dancers return to their original positions using partner switches; same-bird switches; and opposite-bird switches, with the requirement that no transposition can be used twice. We note that this restriction provides a useful visual cue for learning the dance: If you've already switched with a person, you will not switch with that person again until the dance completes.

The Two Step

We'll go through a two-step process to create the dance. First, we'll identify the actual transpositions used. Then we'll convert these into common EC dance figures to produce a danceable choreography.

In principle, the Futurama Theorem could work with any number of couples. For space reasons, we'll create choreograph for a common EC dance configuration: three couples in a line, each dancer facing their partner ("inward"); see Figure 1. We then use the following steps:

- First couple switch places, $(L1\ R1)$.
- Same bird switch, $(L1\ L2)(R1\ R2)$.

Note that at this point, the only remaining move is the opposite bird switch, $(L1\ R2)(L2\ R1)$. However, if these are called, the net effect would be that of couple two switching places, $(L2\ R2)$. So in some sense, calling additional steps would reduce the net movement of the dancers. Since dance is, at its heart, about movement, we regard our current sequence of steps as producing the greatest possible choreographic value.

To return all dancers to their original positions, we bring the third couple into the dance. We'll use the following sequence which, as promised, does not re-use a transposition:

- Opposite bird switch with the first and third couples, $(L1\ R3)(R1\ L3)$.
- Opposite bird switch with the second and third couples, $(L2\ R3)(R2\ L3)$.
- Same-bird switch with the first and third couples, $(L1\ L3)(R1\ R3)$,
- Third couple switch places, $(L3\ R3)$.

To create an ECD from this sequence of transpositions, we'll need to do two things. First, identify the transpositions with dance moves. Second, include additional steps to allow synchronization of the dance with the music.

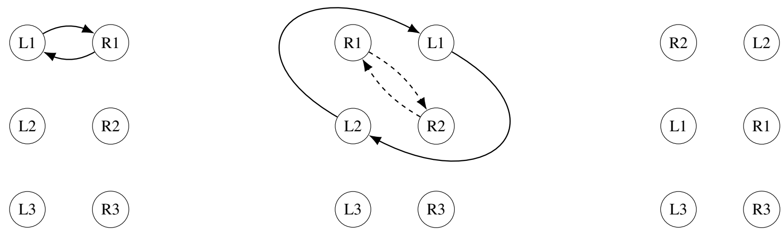

Figure 1: *The First Part of the Dance*

To identify dance moves with transpositions, note that as long as two transpositions don't involve the same dancer, they can be done sequentially or simultaneously. Sequential transpositions pose no special challenges, but simultaneous transpositions need more careful handling to avoid collisions between dancers.

One common approach is a "cast," where one couple splits apart and rejoins in another couple's position, while a second couple moves along a nonintersecting path. Generally speaking, one couple casts out (away from the set) and the other couple moves into the positions vacated by the casting couple.

For music, we'll use Playford's "Upon a Summer's Day" [1]; readers may wish to view [3] to see Playford's dance as originally choreographed. To synchronize the dance with the music, we must introduce some additional steps to mark time within the music. To stay within the constraints of the Futurama Theorem, these steps cannot introduce new transpositions. Thus dancers may move from a position, but must return to that position. A common "null effect" step is a double: two walking steps. Typically these are done in pairs: for example, a double forward, followed by a double back to place.

Another common null step is a set and turn: a dancer steps in one direction, then in the other, then turns in place. Usually these are also done in pairs (see [3] for an example).

"Upon a Summer's Day" also has a chorus, which consists of the top couple (whichever number they are) casting down to the bottom; Playford's choreography has a slight embellishment ([1, 3]). This is done three times, so that at the end of the chorus, all dancers are back at their original positions.

While, strictly speaking, the chorus is a sequence of transpositions, in order to synchronize with the existing music we will treat it as if it is another "null effect" step and include it in the dance.

Our basic choreography will be for a line of three couples, with each Lark facing their partner Robin ("facing into the set"); see Figure 1. The first part of the dance, with the transpositions for reference, will be:

- Larks and Robins forward (towards the top of the page), then double back to place,
- First couple trade places, $(L1\ R1)$,
- Larks and Robins forward, then double back to place,
- Robins trade places (passing through center), $(R1\ R2)$,
- Simultaneously Larks trade places, circling around the outside $(L1\ R1)$,
- Chorus (lead couple casts down to bottom, repeated three times).

For the second part of the dance, we'll use simultaneous transpositions and casting to avoid collisions:

- "Cast and cross": First Couple turn away from each other and cast out around Third Couple's position, cross and end "improper" (Lark on the right side of the set, Robin on the left) in the Third Couple's original position (solid lines in Figure 2). At the same time, the Third Couple crosses and advance into the position vacated by the First couple (dotted lines in Figure 2a); this is $(L1\ R3)(R1\ L3)$.
- "Set and turn"

- Cast and cross, this time with the second and third couples, $(L2\ R3)(R2\ L3)$ (Figure 2b)
- Set and turn,
- Chorus (lead couple casts down to bottom, repeated three times),
- Cast and cross with the first and third couples, $(L1\ L3)(R1\ R3)$ (Figure 2c). Note that the third couple should end improper; also this cast and cross is a same-bird switch, whereas the previous cast and cross moves were opposite-bird switches.
- Set and turn,
- Third couple switch places, ending the dance with all dancers back in their original places (Figure 2d).
- Chorus,

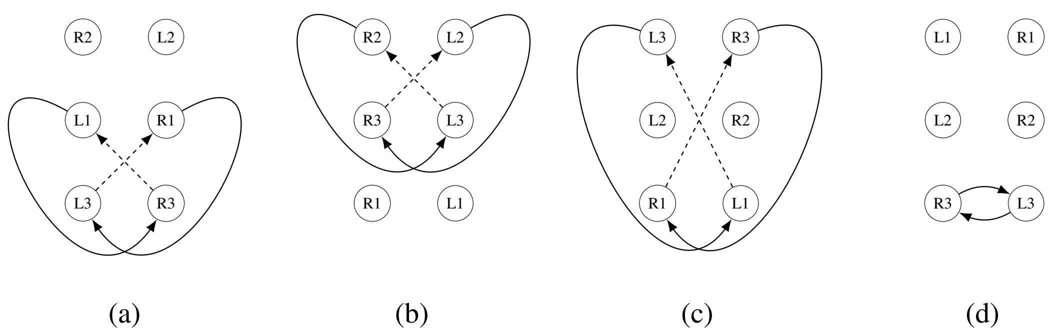

Figure 2: *The Second Part of the Dance*

Summary and Conclusions

This article describes the creation of a base choreography for an English Country dance expressing a novel mathematical theorem. The approach can be extended to larger sets, allowing for more intricate choreographies.

References

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