Cube Compound Puzzles

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Abstract

A compound of n cubes is a non-convex polyhedron formed from the union of n identical concentric cubes. There are many ways to create a compound of n cubes with overall polyhedral symmetry. We consider four particular cases of a compound of n cubes, where n ranges from 2 to 5, inclusive. These polyhedra are dissected into interlocking pieces, making assembly puzzles. We discuss the design of such puzzles.

Introduction

Figure 1 shows four versions of a compound of n cubes, C_{n}, where n = 2, 3, 4 and 5. Each cube is colored with a different color, making the cubes easily distinguishable. These are fascinating geometrical objects, and each also comes apart into pieces. Assembly from a set of pieces is an interesting additional challenge.

Figure 1: *3D printed puzzles in the shape of a compound of n cubes: C_2, C_3, C_4 and C_5.*

In Figure 1, notice the rotational symmetries about the vertical axis of order 6, 3, 4 and 5, respectively (this symmetry is mirrored by the black stands as well). The reader should not be led to assume that the polyhedra shown in Figure 1 are the only cube compounds possible. Many others can be found in [10][13]. These are just four examples which we have converted into mechanical puzzles.

C_2 can be described by starting with two concentric cubes and rotating one by 60° about a 3-fold axis. A wireframe version of this polyhedron appears in Escher's wood engraving *Stars* [8]. C_3 is known from its appearance in Escher's lithograph *Waterfall* [9]. It can be obtained starting from three concentric cubes by rotating each by 45° about each of the three axes of 4-fold symmetry. C_4 was described in 1959 by T. Bakos [1] and is sometimes called Bakos' compound. It can be obtained starting from four concentric cubes by rotating each by 60° about one of the four 3-fold axes. C_5 has the highest degree of symmetry of the four, it has icosahedral symmetry and in addition is vertex-, edge- and face-transitive. The cube corners lie at the vertices of a regular dodecahedron and each vertex is shared by exactly two cubes.

Table 1 lists various properties of these four polyhedra. The intersecting solid is the set of points common to all n cubes. The convex hull is the union of all line segments joining any pair of points. The polyhedron volume is obtained starting with n unit cubes, exact and approximate values are given.

Table 1: *Properties of C_n in Figure 1. The volume is for unit cubes.*

n (cubes and colors)	intersecting solid S_n	convex hull	symmetry type	volume
2	hexagonal dipyramid	elongated hexagonal dipyramid	D6	$^5/_4 = 1.25$
3	chamfered cube	irregular truncated octahedron	octahedral	$\frac{1}{2}(24 - 15\sqrt{2}) \approx 1.39340$
4	small triakis octahedron	chamfered cube	octahedral	$\frac{229}{154} \approx 1.48701$
5	rhombic triacontahedron	regular dodecahedron	icosahedral	$\frac{1}{2}(55\sqrt{5} - 120) \approx 1.49187$

We can consider each puzzle a dissection of C_n. However, most dissections will not make a good puzzle. For a good puzzle, we require that the pieces be *interlocking*, meaning loosely that the pieces hold themselves together, and the assembled puzzle does not fall apart. We'll require something even stronger—namely when the puzzle is assembled, no piece can move (when all other pieces are stationary). The reader may wonder how it is possible for such an object to come apart. The answer is that either the puzzle comes apart in two halves (each half consisting of at least 2 pieces) or disassembly may require that all pieces move simultaneously, a process known as coordinate-motion [4].

In order to gain more insight into the four puzzles in Figure 1, a photograph of each puzzle with one piece removed is included in the supplement. These photos give further insight how the pieces go together to form C_n.

The polyhedra C_n are interesting objects in themselves, why convert them into mechanical puzzles? Because it gives additional insight into the symmetry of these objects, and the assembly process can yield further insight. There is the challenge of figuring out how the pieces assemble into C_n, and the colors must be carefully matched to give the correct color symmetry. Each puzzle can be assembled into the correct shape but with incorrect color symmetry.

Cube Compound Geometry

The notation C_n will be reserved for the specific n cube compound in Figure 1. The avoid confusion, we'll use a different name after conversion into a mechanical puzzle. The puzzle designer often provides a name for their design, for emphasis these puzzle names will be italicized.

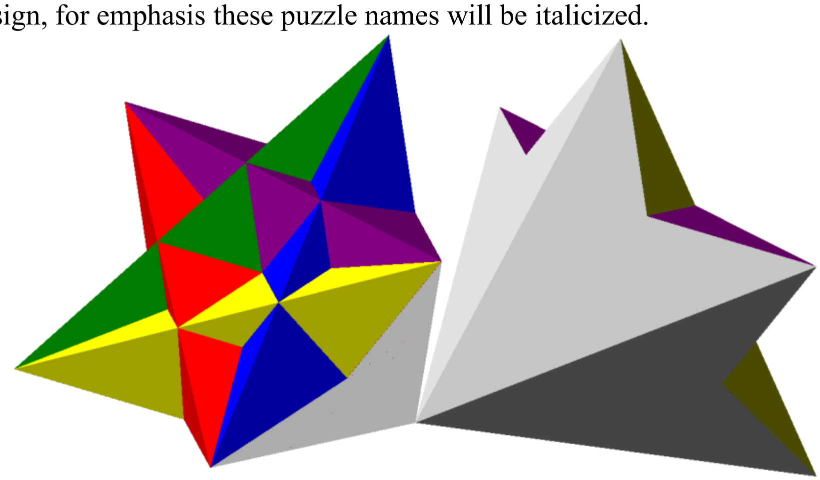


Figure 2: *Face dissection of C_5 by a regular dodecahedron (two of the twelve pieces shown).*

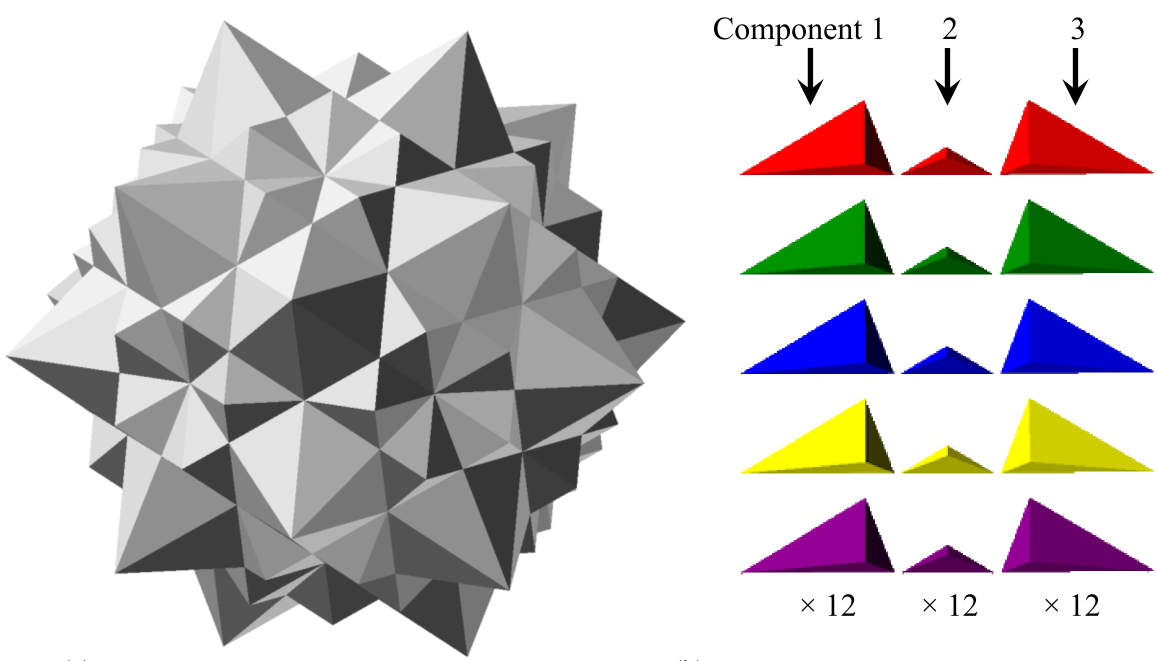
We note that it is relatively easy to use the symmetry of C_n to dissect it into identical pieces which do not interlock. We can accomplish this using a *face dissection*. To perform a face dissection, we require a polyhedron P sharing the symmetry of C_n. As an example we will consider C_5 with P the regular dodecahedron. First, we translate P so that it's center coincides with that of C_5, and scale it up so that it encloses C_5. We then cut C_5 into pieces by cutting along each triangle defined by each edge of P and the center. Figure 2 shows two of twelve identical pieces that result from this process. Here we have aligned the vertices of P and C_5, but this need not be the case.

Note that the number of pieces in the dissection is the same as the number of faces in P. Also note that when we say the two pieces are identical, we refer only to their shapes. The coloring of the two pieces in Figure 2 is not the same. Finally, the pieces are not interlocking. If we build C_5 from the 12 pieces it will come apart easily.

Given a compound of *n* cubes C_n and $1 \le i \le n$, we define $S_i(C_n)$ as the set of all points $x \in \mathbb{R}^3$such that x lies in at least i cubes. Note that by definition $S_1(C_n) = C_n$, and $S_n(C_n)$ is the intersecting solid in Table 1. $S_i(C_n)$ for $1 \le i < n$ can be considered *stellations* of the intersecting solid $S_n(C_n)$. The polyhedra $S_i(C_n)$ for $1 < i < n$ are interesting polyhedra by themselves; each shares the symmetry of C_n.

$S_2(C_n)$ is a special case which we call the *core* of C_n, it is the polyhedron consisting of all points common to two or more cubes. To complete C_n we need add all points which are in exactly one cube. These separate nicely into groups of polyhedra which we call *colored components*. When all colored components are added the core itself will not be visible.

For example, Figure 3 shows the core $S_2(C_5)$. This polyhedron is a stellation of the rhombic triacontahedron with 360 faces. We can complete C_5 by gluing on 180 colored components, which in this case are tetrahedra of three types. A large tetrahedron (component 1), small tetrahedron (component 2) and component 3 which is the mirror image of component 1. We need 12 of each component in each of five colors, so 60 of each component for a total of 180 components. We glue each component to the core. Each component covers 2 faces of the core, so that the core is not visible when all components are in place. This is the reason the core is shown in a neutral color.



(a) (b)

Figure 3: *(a) Core and (b) colored components for C_5. To complete C_5, 12 copies in 5 colors are needed.*

When designing puzzles, it is useful to think of each in terms of the core plus colored components. We consider each puzzle made from p identical pieces. The most difficult design task is to choose p together with the decomposition of the core into p identical pieces. The colored components will be added as a final step, if no component is to be split we must have that p divides the number of colored components of each type. For C_5 this means that p should divide 60.

The final design task is to add the colored components to each piece. In the assembled puzzle the core will not be visible, so we can choose any color for the core.

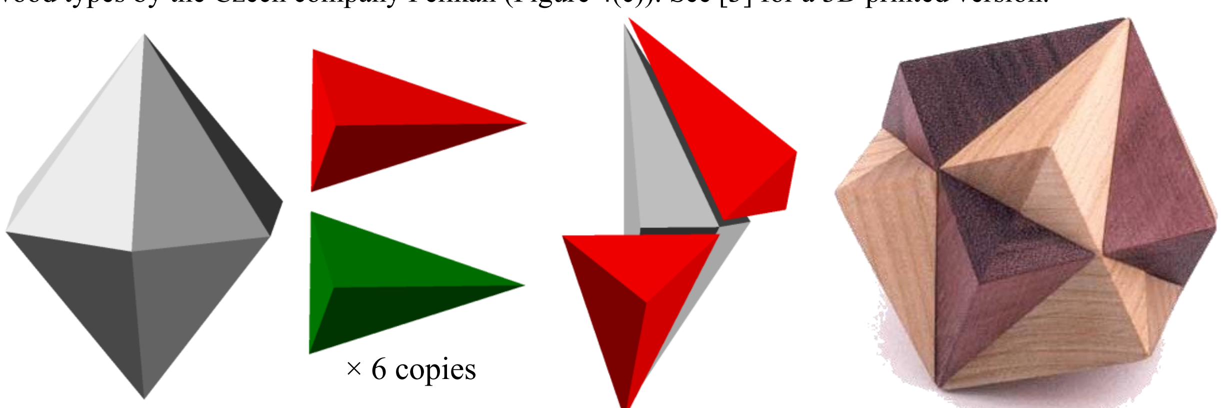
Puzzle Design Details

Table 2 contains details about the four puzzles in Figure 1; most use p identical pieces. When p is even we allow $p/2$ identical pieces plus $p/2$ mirror image pieces; for example in the *Compound of Three Cubes*. The column *assembly* indicates how the puzzle goes together.

Table 2: *Properties of the puzzles in Figure 1.*

n (cubes and colors)	puzzle name	p (no. pieces)	cube size	diameter	assembly	colors/ piece
2	Kubusmix	6	5 cm	8.7 cm	halves	1
3	Compound of Three Cubes	6	6.3 cm	10.9 cm	halves	3
4	Bakos' Puzzle	4	7 cm	12.1 cm	coordinate-motion	4
5	Compound of Five Cubes	10	8.1 cm	14.0 cm	coordinate-motion	5

The core for C_2 is a hexagonal dipyramid, Figure 4(a). To complete C_2 we must add six copies of the colored component in two colors. We dissect the core into six identical pieces by slicing it vertically like a pie into six 60° sections. This can also be described as a face dissection using a hexagonal prism of infinite length. Two colored components are then added to form the basic piece, as shown in Figure4(b). Since the two components have the same color, each piece is printed in a single color, three red and three green. This puzzle was invented around 2002 by Rik Brouwer; he called it *Kubusmix* [6], around 100 were made in two wood types by the Czech company Pelikan (Figure 4(c)). See [3] for a 3D printed version.



(a) (b) (c)

Figure 4: *Converting C_2 into a puzzle.*

An interesting variation to *Kubusmix* is to glue the pieces together in pairs. The resulting three identical piece puzzle can only be assembled using coordinate-motion.

The core for C_3 is shown in Figure 5(a). To complete C_3 we add eight copies of two types of colored components in three colors, Figure 5(b). The puzzle piece is created by dissecting the core into six identical pieces, then adding 2 large components and 4 small components to make the basic piece, shown in Figure 5(c). If the dark grey pyramid in Figure 5(c) is included with the piece the assembled puzzle will be solid. In all cases below this dark grey pyramid is removed and the puzzle contains a hollow cubical void.

To complete the puzzle, we need to add 12 large components in three colors. There are 4 places on each piece these components could be added. There are many options available to complete this puzzle, depending on where these 12 components are added. We can even find a version with six different pieces.

If we want identical pieces, our options are reduced. Rik Brouwer used six copies of the piece in Figure 6(a), he calls this puzzle *Trikube* [3]. This puzzle was also sometimes made by cutting each piece into two identical parts for a total of 12 pieces. Theo Geerinck and Symen Hovinga used six copies of the piece in Figure 6(b), they call this puzzle *Triplicato* [11]. I made the hybrid piece in Figure 6(c), each piece now has all three colors, but we require three identical pieces and three mirror image pieces.

All of these versions come apart in halves. The difficulty varies, however, due to the stability of the two halves. *Triplicato* is the most frustrating of the three, as the halves are very unstable.

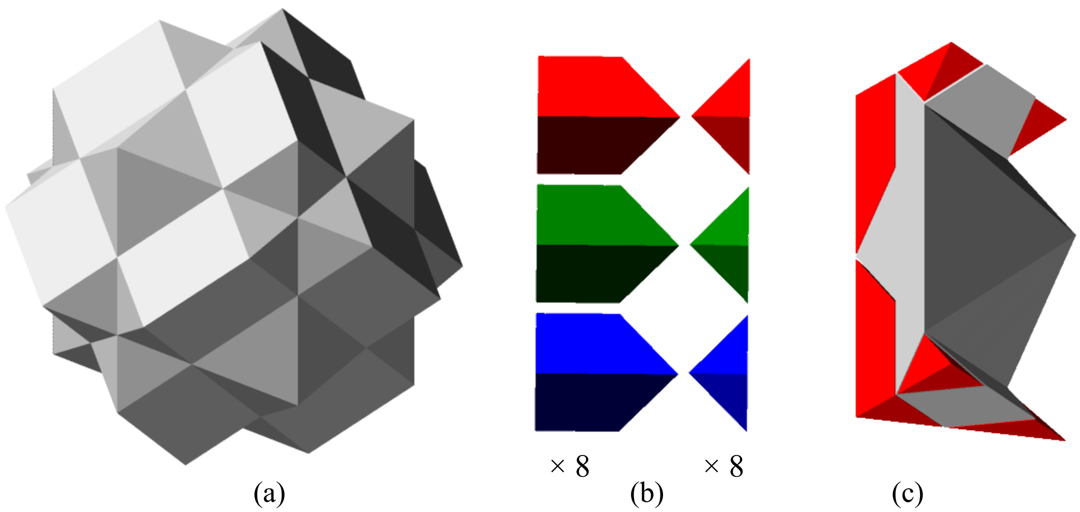


Figure 5: *Converting C_3 into Bakos' Puzzle.*

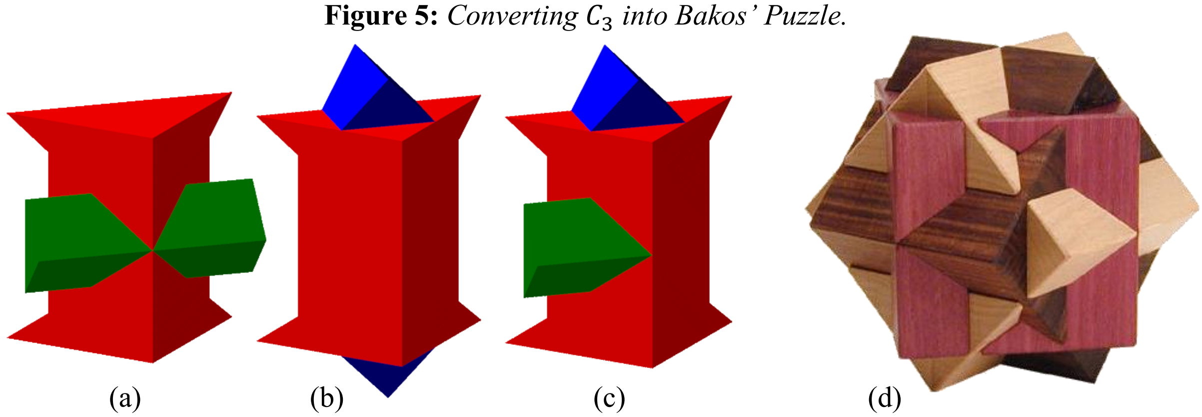
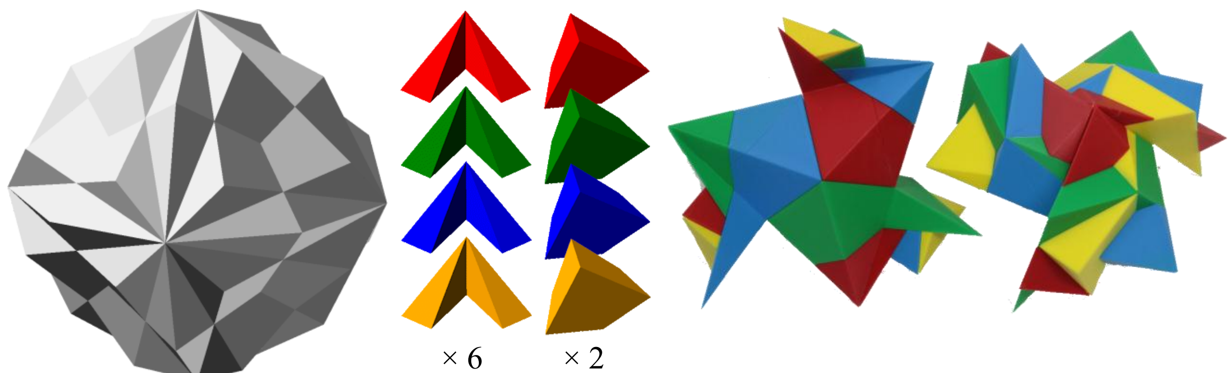


Figure 6: *Three options for the C_3 puzzle piece: (a) Tricube, (b) Triplicato, (c) Compound of Three Cubes, (d) Trikube made in three wood types.*

The core for C_4 is shown in Figure 7(a). To complete C_4 we must add the colored components shown in Figure 7(b). The fact that there are 8 colored components of the second type implies that the number of pieces p should evenly divide 8. This suggests that p should be 4 or 8.

C_4 and its core have four clear axes of 3-fold symmetry, like the rhombic dodecahedron. I knew of a dissection of the rhombic dodecahedron into four identical pieces which assemble using coordinate-motion. To create the puzzle piece, I first added all eight of the second colored components to the core, and then intersected it with the rhombic dodecahedron piece. The V-shaped colored components are then added, six to each piece. The resulting puzzle piece is shown in Figure 7(c), I call this *Bakos' Puzzle* [3], the assembled version is in Figure 1. The supplement contains much more detail on the construction of *Bakos' Puzzle*.



(a) (b) (c)

Figure 7: *Converting C_4 into a puzzle.*

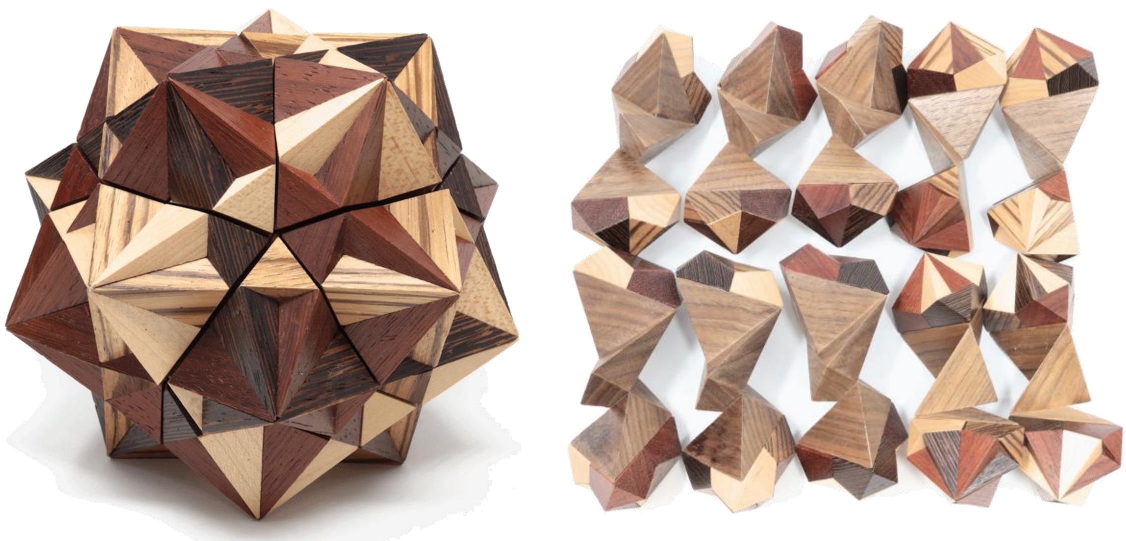


Figure 8: *A wood puzzle of C_5 made by Wayne Daniel. It is made from five I and five J pieces.*

In the collection of Stan Isaacs I noticed a wood puzzle in the shape of C_5 (Figure 8). This remarkable puzzle was made by Wayne Daniel more than 20 years ago using 180 precisely wood components in six wood species. 18 wood components were then glued together to make each of the ten pieces. The assembled appearance is of five intersecting cubes, each composed of a different wood species. The sixth wood type is used by the icosahedron core and is not visible in the assembled puzzle. This puzzle is made from five identical pieces and five mirror image pieces—we now go over its design in detail.

Wayne Daniel used a regular icosahedron for the core of C_5, and his puzzle begins as a 10-piece dissection of an icosahedron [7]. Figure 9(a) shows a face dissection of an icosahedron into 20 "face tetrahedra", one for each face of the icosahedron. Each face tetrahedron is then divided into an inner and outer tetrahedron as shown in Figure 9(b). In his puzzles, Wayne Daniel always used $\theta = 0°$. He then made a puzzle piece using four connected faces of the icosahedron. The puzzle piece is made using the outer, inner, inner, and outer tetrahedra, respectively, from these connected faces.

It turns out a total of ten different pieces can be generated in this fashion, plus their mirror images. These he labeled A-T, with B being the mirror image of A. For our purposes we prefer identical pieces, so which of A-T can make an icosahedron from 10 copies? It turns out that only I, J and M, N can do so [2]. The pieces J and N are shown in Figure 10(a). Note that I, J (and M, N) are mirror image pairs.

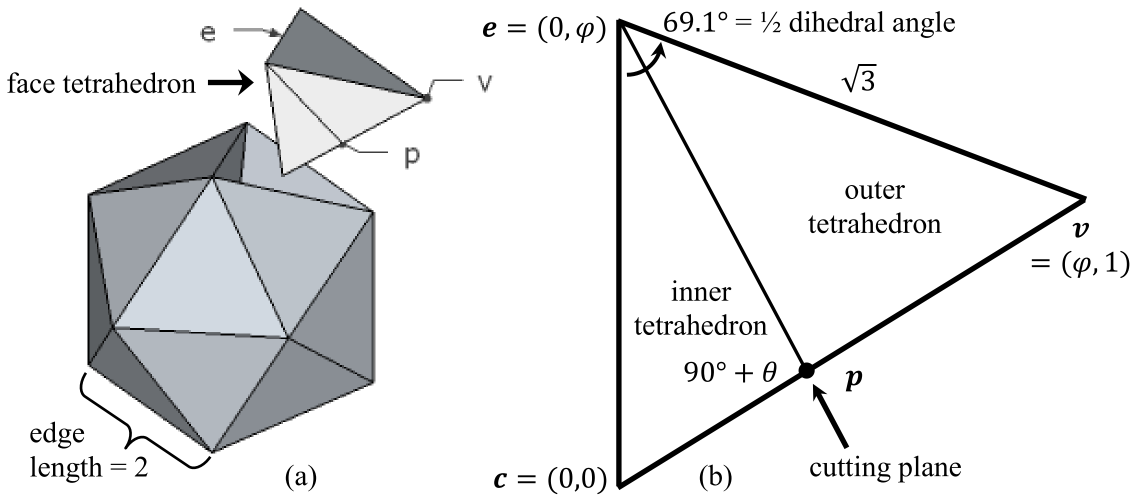


Figure 9: *Icosahedron dissection: (a) the face dissection, (b) details of the cut, φ is the golden ratio.*

A major problem is that although ten J pieces can form an icosahedron, the pieces cannot be assembled! This has not been proven mathematically, but was determined by Wayne Daniel after making the pieces and finding that he could not assemble them. Instead, he used 5×I and 5×J in his C_5 puzzle (Figure 8). This combination of pieces assembles into an icosahedron, when $\theta = 0°$.

Some years later the woodworker Stephen Chin discovered if he increased slightly the value of θ, the 10×J puzzle could be assembled [2]. The best value of θ is determined by trial and error, and every trial involves making a complete set of 10 pieces and trying to assemble them into an icosahedron.

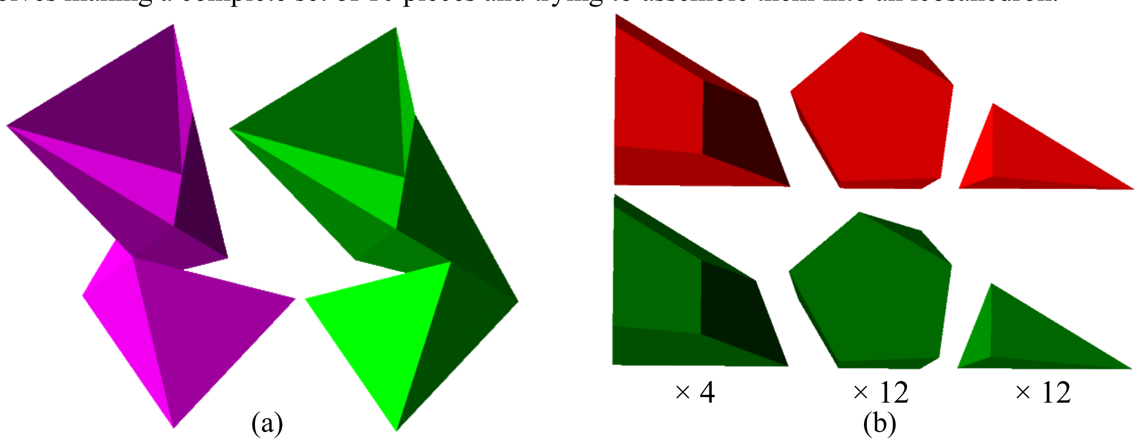
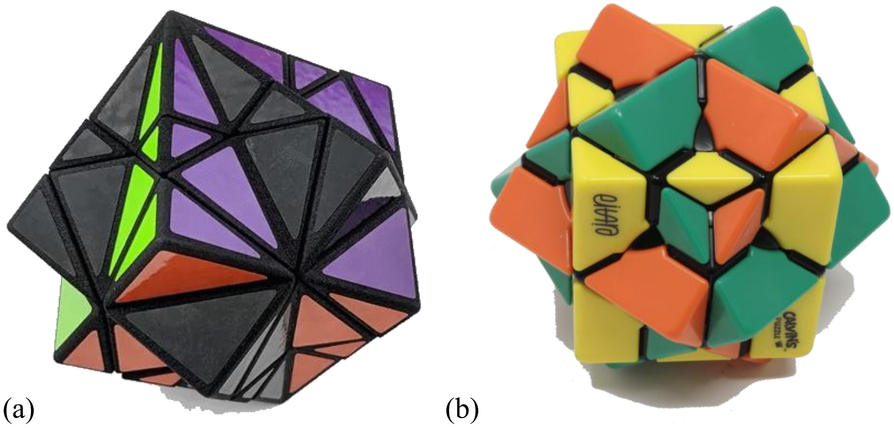


Figure 10: *(a) Pieces J and N with $\theta = 0°$; (b) colored components to make C_5 with an icosahedron as the core. Note that components in all five colors are needed; only red and green are shown for clarity.*

The final step is to convert the 10 piece icosahedron puzzle so that the outer shape is C_5. Since the core is an icosahedron, and not the polyhedron in Figure 3(a), the colored components are modified from those in Figure 3(b)—they are shown in Figure 10(b). Note that Component 3 is the only one which is unchanged from Figure 3(b). 14 components are glued to each J piece, great care must be taken to ensure the colors are correct. The 3D printed C_5 puzzle in Figure 1 uses 10×J and $\theta = 7.65°$. See [5] for plans to make your own 3D printed copy of *A Compound of Five Cubes*.

Alternative Puzzles

There are other ways to convert C_n into a mechanical puzzle. An alternative concept is to make C_n into a twisty puzzle like *Rubik's Cube*. These puzzles do not come apart, but the coloring changes after twisting parts of them along certain axes. To solve the puzzle the original coloring must be restored. C_2 and C_3 have been made into twisty puzzles, see Figure 11.



(a) (b)

Figure 11: *Twisty puzzles based on C_2 and C_3: (a) Conjoined Twins by David Pitcher [12] (b) Eitan's Tricube by Eitan Cher.*

Summary

We have taken four specific versions of a Compound of n Cubes and detailed how we convert each into a mechanical puzzle. The pieces are either identical in shape or there are $n/2$ identical pieces and $n/2$ identical mirror image pieces. No piece can move by itself when the puzzle is assembled. Many of the puzzles were designed previously, and several have been made from wood. We use 3D printing to produce modern versions.

Acknowledgements

I thank Rik Brouwer for generously sharing details of his puzzle designs, and Stan Isaacs for lending me the rare Wayne Daniel puzzle in Figure 8. I thank the anonymous referees for many helpful comments.

References

[1] T. Bakos. "Octahedra inscribed in a Cube." Mathematical Gazette, Vol. 43, pp. 17-20, 1959.

[2] George Bell. "More Icosahedron Puzzles." Cubism For Fun #87, March 2012, pp. 10–15, http://www.gibell.net/puzzles/CFF/Bell2012_MoreIcosaPuzzlesCFF87.pdf

[3] George Bell. Cube Compound Puzzles, https://www.printables.com/@GBell/collections/2045013

[4] George Bell. "An Initial Attempt at a Mathematical Treatment of Translational Coordinate-Motion Puzzles." *Bridges Conference Proceedings*, Richmond, Virginia, USA, 1–5 Aug., 2024, pp. 187-94. https://archive.bridgesmathart.org/2024/bridges2024-187.html

[5] George Bell. PolyPuzzles, https://www.etsy.com/shop/PolyPuzzles

[6] Rik Brouwer. Kubusmix, https://www.johnrausch.com/DesignCompetition/2002/

[7] Wayne Daniel. "Some Icosahedron Puzzles." Cubism For Fun #50, Part 3, October 1999, pp. 13–17.

[8] M. C. Escher. *Stars*. wood engraving, 1948.

[9] M. C. Escher. *Waterfall*. lithograph, 1961.

[10] George Hart. https://www.georgehart.com/virtual-polyhedra/compound-cubes-info.html

[11] Symon Hovinga. https://www.treatstock.com/3d-printable-models/5151270-triplicato

[12] David Pitcher. Conjoined Cubes, http://puzzleworld.org/DesignCompetition/2019/

[13] Hugo F. Verheyen. *Symmetry Orbits*. Birkhauser, 1996.