Picturing Automorphisms of the Fano Plane

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Abstract

The Fano plane is an arrangement of 7 points on 7 lines. An automorphism of the Fano plane corresponds to permuting the points, while conserving the relation between them. These automorphisms correspond to triangles in a tiling of the Klein quartic. Moving through all the automorphisms is achieved by following a Hamiltonian path through a Cayley graph of the automorphism group, which can be represented on the Klein quartic. The results are an artistic illustration of an automorphism, and an animated walk through an automorphism group.

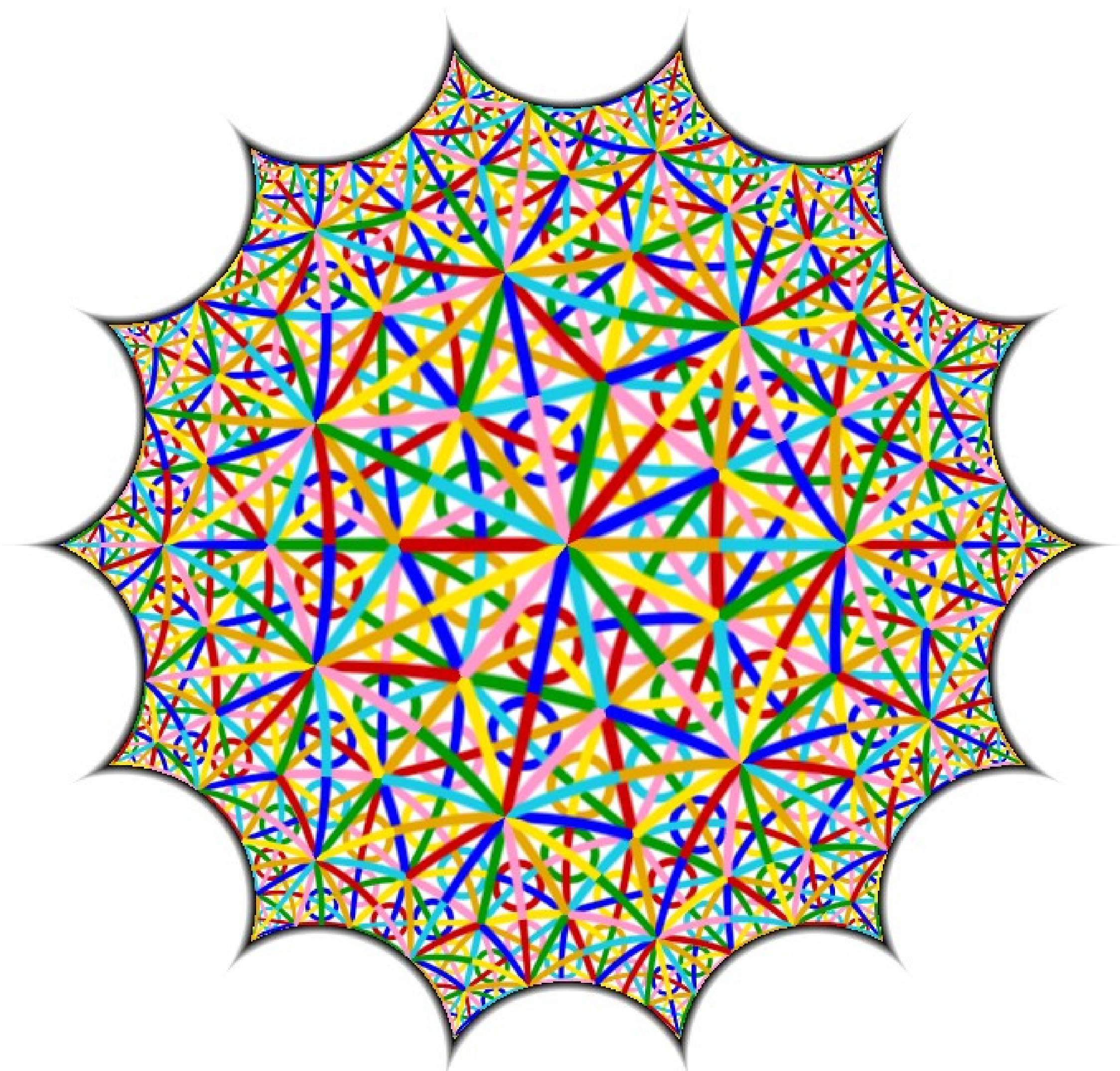
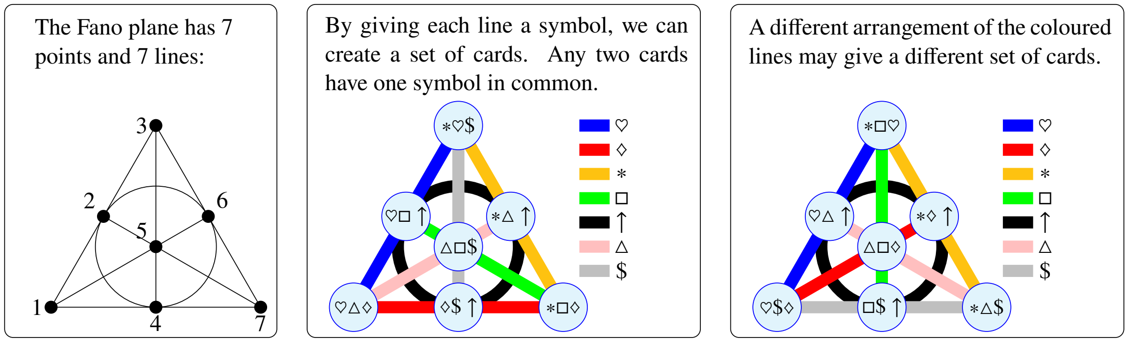


Figure 1: *Automorphisms of the Fano plane placed on the Klein quartic.*



(a) **(b)** **(c)**

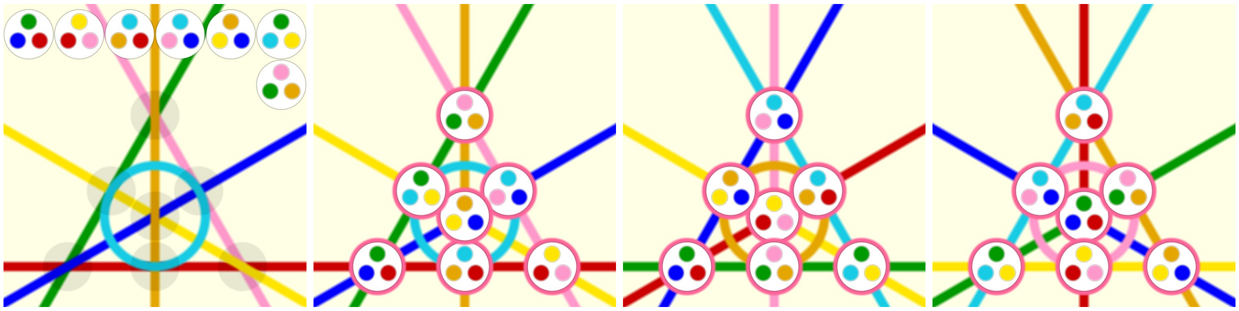
Figure 2: *The Fano Plane and "Mini Dobble". "Dobble" is a card game also known as "Spot it!" [3].*

The Fano plane

We can define the Fano plane to be the set $P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ together with the set $L = \big\{\{p_1, p_2, p_3\}, \{p_1, p_4, p_7\}, \{p_3, p_6, p_7\}, \{p_3, p_4, p_5\}, \{p_2, p_5, p_7\}, \{p_1, p_5, p_6\}, \{p_2, p_4, p_6\}\big\}$. A definition in a more general framework is given in [12, §2.2]. Abstractly, this is a set P and certain subsets of P. We interpret these geometrically, as P being a set of "points", and L being a set of "lines", as in Figure 2 (a). We say that two "lines" intersect if they have a "point" in common, e.g., the "lines" $\{p_1, p_2, p_3\}$ and $\{p_1, p_4, p_7\}$ both contain p_1, and so we say they intersect in p_1. Figure 2 (a) is a geometrical representation of the Fano plane; we draw dots for each element of P, which are labeled in the diagram with the numbers i for p_i, for i from 1 to 7. Actual lines (not just sets of three points) are drawn to indicate the "line" relationships between the points. There are many possible ways to represent the Fano plane as a drawing. As well as topologically distorting the picture to convey the same relationships, we could also cut some lines, e.g., the arc from p_4 to p_6 could be removed, leaving an arc passing through p_2, p_4, p_6, which still conveys the relationship between these points, as being in the common set $\{p_2, p_4, p_6\}$ in L. Although the Fano plane is a combinatorial object, we will refer to the representative drawing in Figure 3(a), with any choice of colouring of lines, as the Fano plane. The Fano plane can be used to answer the question "can you find a set of cards, with three different features on each card, such that any two cards have exactly one feature in common?" To use the Fano plane to solve this problem, associate a symbol to each line, and a card to each point, as in Figure 2 (b). Place three symbols on each card, determined by the symbols of the line on which the point lies. The same set of cards can be obtained from different colourings of the lines, as in Figure 3, where we now use dots of given colours, rather than symbols. Figure 3(a), taken from [14], shows the lines with the corresponding cards waiting to be placed in the right places. Figure 3(b) shows the cards placed correctly; (c) and (d) show other colourings of lines resulting in the same cards. This is related to the game "Spot it!" [3] [19] and to art work in [9].

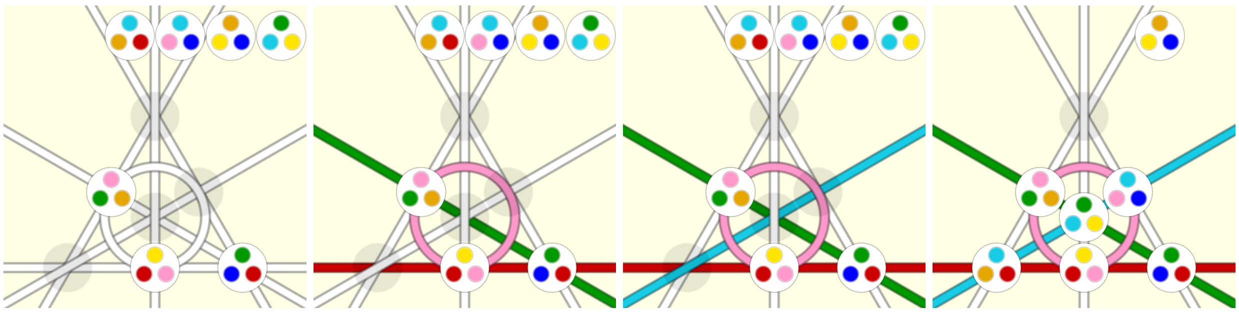
Not all arrangements of the same colour lines give the same set of cards, as in Figure 2. How many colourings of lines *do* give the same set of cards? This paper describes how to find an artistic way to animate and display the solution to this problem. The resulting animation [16] was achieved by placing the different coloured Fano planes on the Klein quartic, as in Figure 1.

I use the terms *arrangements* and *colourings* interchangably. Since e.g., Figures 2 (b) and (c) differ by either moving the lines in the picture to different positions, or keeping the lines fixed, but changing their colour. By an *automorphism* of the Fano plane I (essentially) mean a different way of colouring the lines which gives rise to the same set of cards. These are permutations, of either lines or colours. It turns out there are 168 possible automorphisms of the Fano plane [20]. To check this, we explain how any automorphism is determined by choosing the positions of 3 non-collinear points (i.e., they are not all in the same line).



(a) **(b)** **(c)** **(d)**

Figure 3: *Arrangement of the Fano plane with the same resulting card sets [14].*



(a) **(b)** **(c)** **(d)**

Figure 4: *Determining an arrangement of the Fano plane given the placement of three non-collinear points.*

Such a set of points can be considered as a "basis" (in a very concrete way, these correspond to a basis of three dimensional space consisting of vectors with all entries 0 or 1; the automorphism group is naturally isomorphic to the group $\mathrm{PGL}(3, \mathbb{F}_2)$). This is illustrated in Figure 4. We can take our "basis" to be the cards $(B, R, G), (G, P, O), (Y, R, P)$. Here, for short, (B, R, G) and so on, mean the card with blue, red, green dots, for example. Our colours are:

B: blue; G: green; Y: yellow; O: orange; R: red; P: pink; C: cyan.

We now construct an automorphism. We can place the card (B, R, G) in any of 7 positions; then (G, P, O) can be placed in any of the remaining 6 positions. Now there are 5 places left, but since (Y, R, P) is not on the same line as both (B, R, G) and (G, P, O), we have 4 choices for the position of (Y, R, P). The remaining card positions are all determined by the three "basis" cards. For example, in Figure 4(a), the non-collinear positions of these three cards are chosen. This determines the colours of lines through pairs of them (b). There is only one line not passing through any of them, which must be coloured with the colour not appearing on any of the three cards (c). Once this line is coloured, this determines the colours of all the points on it, and then there is only one card left to place (d). This gives a total of $7 \times 6 \times 4 = 168$ possible arrangements of the points. A program goes through these in this order [14]. I.e., Fix the position of (B, R, G), and try all the possible positions for (G, P, O). For each of these positions, go through all the possible positions of (Y, R, P). Aesthetically, I wasn't happy with the sequence, because in moving from one arrangement to another, we have a jumble of different transformations. I prefer a more regular way to move through the arrangements, as though following a dance pattern, with points being the dancers. In many traditional dances, when you are at position X you always move next to position Y, or at least have a limited number of moves. The regularity and limited nature of options gives a more pleasing structure to the motion.

It is well known that the automorphism group of the Fano plane is isomorphic to the automorphism group

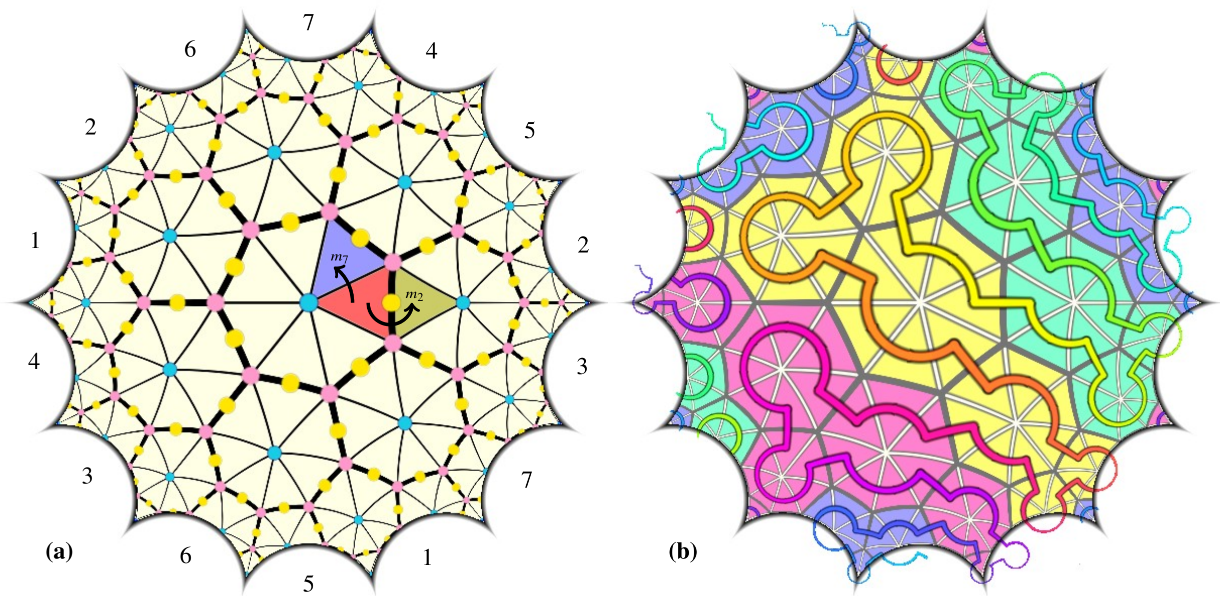


Figure 5: *Representation of the Klein quartic. (a) Generating rotational symmetries indicated. (b) Pair of pants decomposition and Hamiltonian path on Cayley graph indicated.*

of the Klein quartic. This has been used artistically, for example, in [6]. We will use this correspondence to find a nice way to move through the automorphisms of the Fano plane.

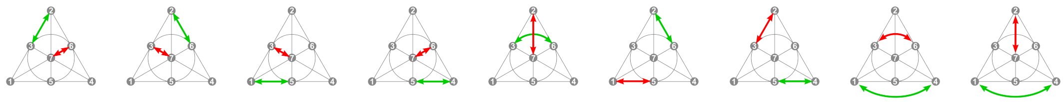
The Klein Quartic

There are many descriptions of the Klein quartic [5]. Figure 5 shows a 14-gon which can be identified with the Klein quartic. The edges with the same label are glued together to form a surface, which can be tiled by 24 heptagons. This leads to beautiful arts works, for example in [8], [7], [13].

The automorphisms of the Klein quartic, which are "more or less" rotations, each corresponds to a triangle in a tiling of the space. Fix a "base" triangle – I choose the red triangle in Figure 5(a). Each automorphism will map this triangle to one of the 168 triangles, and is completely determined by which triangle we choose to map it to. Here a *map* is a set of instructions for pairing up the elements of one set with elements of another set. The automorphisms of the Klein quartic are generated by two rotations, denoted m_7 and m_2, as in Figure 5. In this figure, each cyan point is the centre of an order 7 rotation, each yellow point is the centre of an order 2 rotation, and each pink point is the centre of an order 3 rotation. The composition $m_2 m_7^{-1}$ results in an order 3 rotation. Combining these rotations, about the pink, yellow, or cyan points, allows us to move the red triangle to any of the other triangles in the tiling. Note that this is a picture in "hyperbolic space" [12, chapter 25], which is a surface which must be distorted to be viewed on the plane, and so the triangles are represented in different sizes, though they should be thought of as being the same size. There are other more mysterious symmetries of the Klein quartic, described, e.g., in [4].

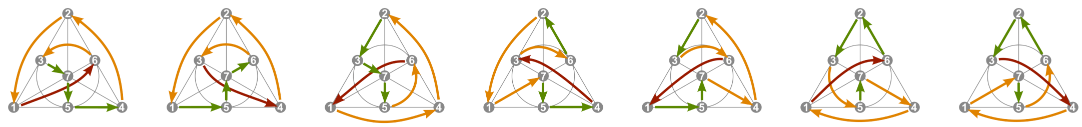
Isomorphisms and Automorphisms

There are 168 automorphisms of both the Fano plane and the Klein quartic. Since these numbers are the same, and since the Fano plane is represented as a triangle structure, we can place all the Fano plane arrangements on the triangles of the Klein quartic. These sets of automorphisms are "groups" in the sense that their



(1) (2) (3) (4) (5) (6) (7) (8) (9)

Figure 6: *A selection of Fano automorphisms or order* 2 *[10].*



(1) (2) (3) (4) (5) (6) (7)

Figure 7: *A selection of Fano automorphisms or order* 7. *[10]*

elements can be composed, and obey certain rules [1, chapter 2]. An *isomorphism* between two groups is a map which respects the group structure, preserving the relations between the elements [1, chapter 7]. If we use an isomorphism to pair up permutations of the Fano plane with triangles on the tiling of the Klein quartic, the relationships between adjacent triangles (how we get from one to another by a rotation) must correspond to the relationships between the chosen Fano arrangements (how we permute the points to get from one arrangement to another).

Figures 6 and 7 are taken from the complete list of automorphisms of the Fano plane at [10]. To find an isomorphism between our groups we need to choose elements of both groups which satisfy the same relations, and match them up. I.e., I want to colour my "base" triangle (the red one in Figure 5) with my starting Fano arrangement (the first one in Figure 8). Then I want to apply m_7 to the triangle, which rotates through $2\pi/7$ counter clockwise, and simultaneously, I want to apply an order 7 permutation of the Fano plane to the arrangement on the triangle. Similarly m_2 will be matched with an order 2 permutation of the Fano plane. The order of a permutation means the number of times it should be applied before getting back to the starting arrangement. For example, we can observe that the sequence of arrangements in Figure 8, which differ by repeated application of a fixed order 7 permutation, are placed on the triangles round the central point in Figure 1, which differ by repeated application of a fixed rotation of order 7.

For a "nice" arrangement of Fano planes on the Klein quartic, I take automorphisms which "match up along edges". I.e., I would like to chose order 7 permutations which map the right side line to the left side line, as in (c) and (d) in Figure 3, and in the sequence in Figure 8. All such order 7 permutations are shown in Figure 7. Similarly, we want to map m_2 to an order 2 permutation which fixes the bottom edge. These are shown in Figure 6. As mentioned, $(m_2 m_7^{-1})^3 = I$ (the identity, the "do nothing" transformation). The pairs of permutations from the lists in Figures 6 and 7 for which this holds are:

[1, 5], [1, 6], [2, 3], [2, 4], [2, 7], [3, 1], [3, 5], [4, 2], [4, 4], [5, 1], [5, 2],
[6, 2], [6, 6], [7, 1], [7, 3], [7, 7], [8, 3], [8, 4], [8, 5], [9, 6], [9, 7].

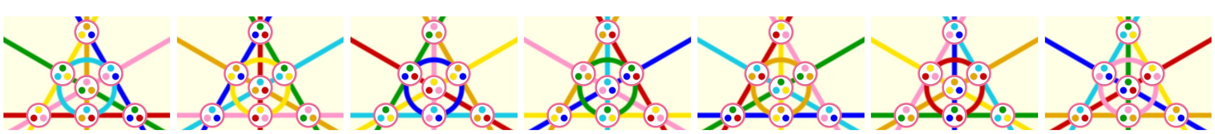


Figure 8: *Repeated application of permutation (5) from Figure 7.*

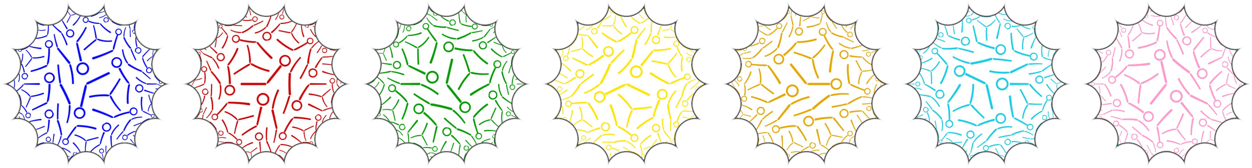


Figure 9: *Figure 1 decomposed into constituent colour components.*

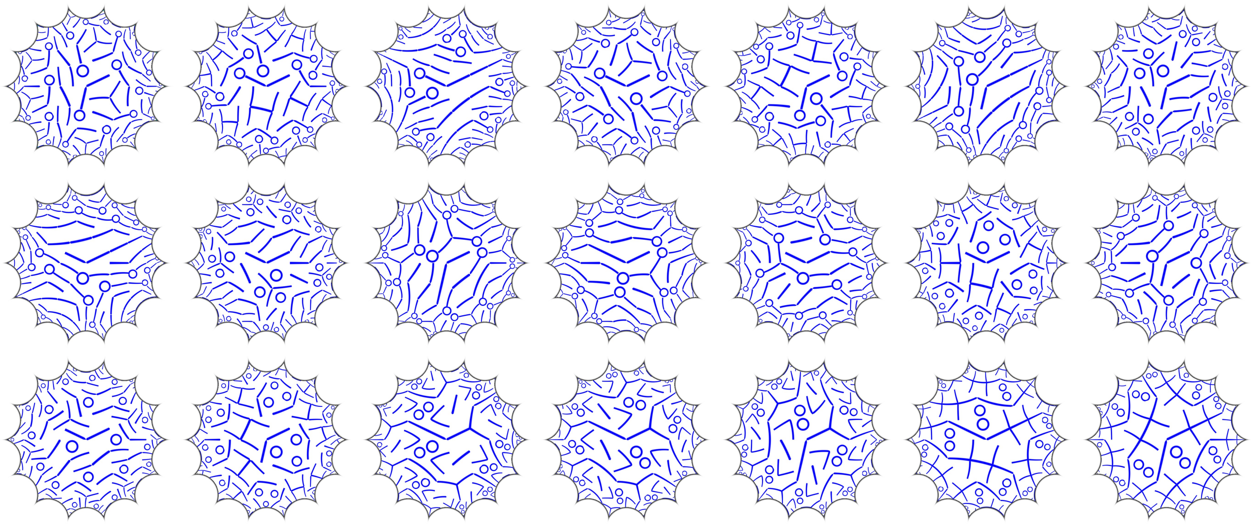


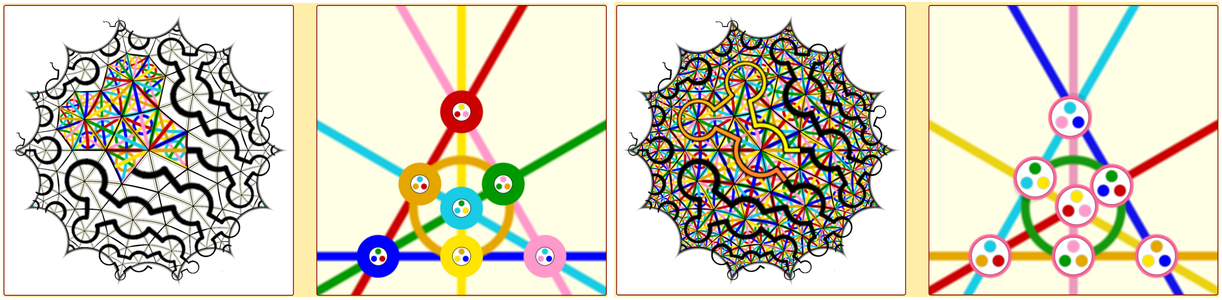
Figure 10: *Blue components only of Fano automorphisms on the Klein quartic.*

The numbers indicate which order 2 permutations to use together with which order 7 permutation, e.g., [8, 3] means the eighth permutation in Figure 6 together with the third permutation in Figure 7 form a compatible pair relative to matching edge colourings of the triangulation of the Fano plane. Notice that each tiling by the Fano arrangements can be decomposed into the constituent colours, which differ only by rotation about the centre point. An example is shown in Figure 9, which shows the colour decomposition of the arrangement in Figure 1. Figure 10 shows the blue components of each of these cases. This is interesting as generative art, and also helps us to understand what we are seeing in Figure 1.

A Hamiltonian Path Through the Cayley Graph

A Cayley graph of a group, corresponding to a choice of generators, is a collection of points and edges where the points corresponds to elements of the group, and the edges, which are lines drawn between the points, correspond to elements being related by generators [18]. Beautiful art works can be created from Cayley graphs [2]. For the automorphism group of the Klein quartic, taking generators m_7 and m_2, we have a Cayley graph as in Figure 12. The gray circles in the middle of each triangle represent the vertices of the graph, corresponding to the group elements, a few of which are labeled. The identity is marked by I. Operations are applied from right to left, e.g., $m_7 m_2$ means first apply m_7, then m_2, starting from the triangle marked I. The blue lines are the edges, corresponding to m_7 or m_2. Wedd [17] gives a different Cayley graph of this group.

In order to pass through all Fano arrangements exactly once, using only permutations corresponding to m_7, m_2, m_7^{-1}, we must draw a Hamiltonian path on the Cayley graph. A Hamiltonian path on a graph is a path through the graph, traveling along the edges, visiting each point exactly once. I have used a "pair of pants" decomposition of the Klein quartic to help find a Hamiltonian path. Using "pants" is a common way to



(a) **(b)**

Figure 11: *Example frames from animations through the Fano plane automorphisms.*

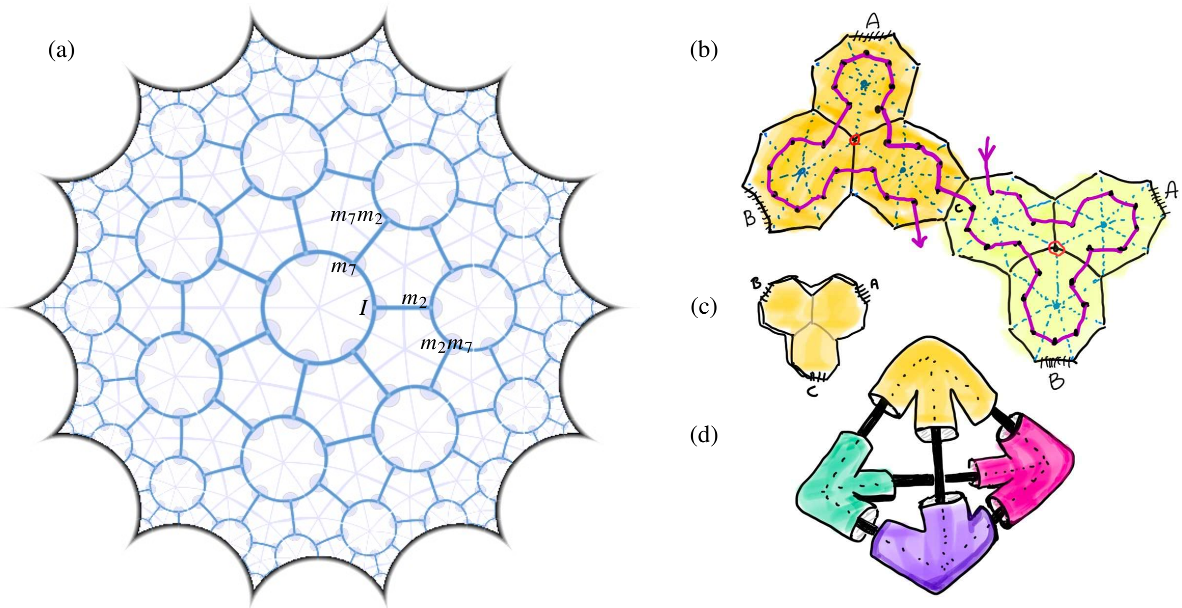


Figure 12: *(a) Cayley graph on the Klein quartic. (b) (c) (d) Pants decomposition of the Klein quartic.*

decompose topological surfaces [11, §9.7]. By breaking into several pieces it is easier to find a Hamiltonian path by finding a path through each pair of pants separately. Such a decomposition is shown in Figures 5 (b), where each pair of pants has a different colour, and comprises 6 heptagons, 3 for the front, 3 for the back. Figure 5 (b) shows one way of gluing the pants together, but the pant structure is not clear. In Figure 12 (b), a single pants component is shown in yellow. In (c) the pants are shown with edges sewn together to look more pant like. In (d), the pants are deformed to fit over the vertices of a tetrahedron. A final gluing of edges will give the Klein quartic (up to topological deformation) [4]. These figures also show a Hamiltonian path. This path goes through each pair of pants one at a time. The crossings of the path from one pair of pants to another lie on the path of the "eightfold way" [5]. The Hamiltonian path can be used to create an animation passing through all the Fano configurations one by one [15]. I have illustrated the correspondence between the Fano plane and the triangles in the Klein quartic, and the Hamiltonian path through these, in various ways. The screenshot in Figure 11 (a) is from [16] where triangles in the Klein quartic are coloured as they are achieved in the Fano plane on the right. In Figure 11(b) all Fano arrangements are shown on the Klein quartic from the beginning, and a path is shown snaking its way round as the Fano pattern appears on the right.

Conclusion

The motivation for this work was the question of how to pass through a list of colourings of the Fano plane (complete, in the sense described in the paper) in a pleasing way, with resulting animation [16]. To achieve the goal, I placed the arrangements on the Klein quartic. Choosing tiles to have matching colour edges is a visual aesthetic choice, which is not necessary for the aesthetics of the motion, but provides an additional constraint, to help choose the animation sequence. In [15] many colour options are possible. Artistically, I prefer two colour hues for the lines, e.g., pinks and greens. However, for a clearer view of the mathematical structure, in this paper I chose bright, clearly distinguishable colours. The many options available at [15] to change parameters are partly for different aesthetic outcomes, and partly to explain different mathematical concepts. There is a tension between these sometimes not entirely compatible goals. Since the actual art works are programmed in WebGL, which is relatively basic, I make decisions such as how hard to make the gradient transition at edges of lines, and so on; that is a question of the medium used to convey this art work; the concept could be realised in different mediums; a future project.

Acknowledgements

I apologise to the mathematician readers for the lack of details, and to the non-mathematician artist readers who might find slightly too much jargon; there is perhaps a tension between writing something suitable for both audiences. I would like to thank the referees for their suggestions for helping to bridge the gap.

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