How N. G. (Dick) de Bruijn Connected Mathematics and the Arts

Tom Verhoeff

Department of Mathematics and Computer Science, Eindhoven University of Technology, Netherlands; T.Verhoeff@tue.nl

Abstract

I explore the ways in which Dutch mathematician Dick de Bruijn connected mathematics and art, on purpose and incidentally, from my perspective as former student and colleague of Dick at TU Eindhoven.

Introduction

Nicolaas Govert (Dick) de Bruijn (The Hague, 9 July 1918 – Nuenen, 17 February 2012; Figure 1, left) was a Dutch mathematician —a mathematicus universalis (one of the last)— and an extraordinary person. In 1946, he was appointed Full Professor at TU Delft, then moved to the University of Amsterdam in 1952, and in 1960 settled at TU Eindhoven. He made original contributions to many fields, in particular, number theory (de Bruijn–Newman constant, Dickman–de Bruijn function, Moser–de Bruijn sequence, de Bruijn–Good Theorem), analysis (de Bruijn's methods in asymptotics), combinatorics (de Bruijn sequence, de Bruijn graph, de Bruijn torus, de Bruijn–Erdős Theorem (there are even two of them), the BEST Theorem (where B stands for de Bruijn), de Bruijn's theorem on filling boxes with bricks, de Bruijn–Klarner Theorem, de Bruijn's generalization of Pólya's Enumeration Theorem), and logic and computer science (de Bruijn indices, de Bruijn notation, de Bruijn factor, Automath, Curry–Howard–de Bruijn correspondence). His algebraic approach to aperiodic tilings is harder to classify (it involves geometry, algebra, combinatorics, topology, crystallography). Also noteworthy are his contemplations on and models for human memory, consciousness, and thinking. All of his scientific work is well documented. In this article, I will explore how de Bruijn connected mathematics and the arts, in ways that may be less well known.

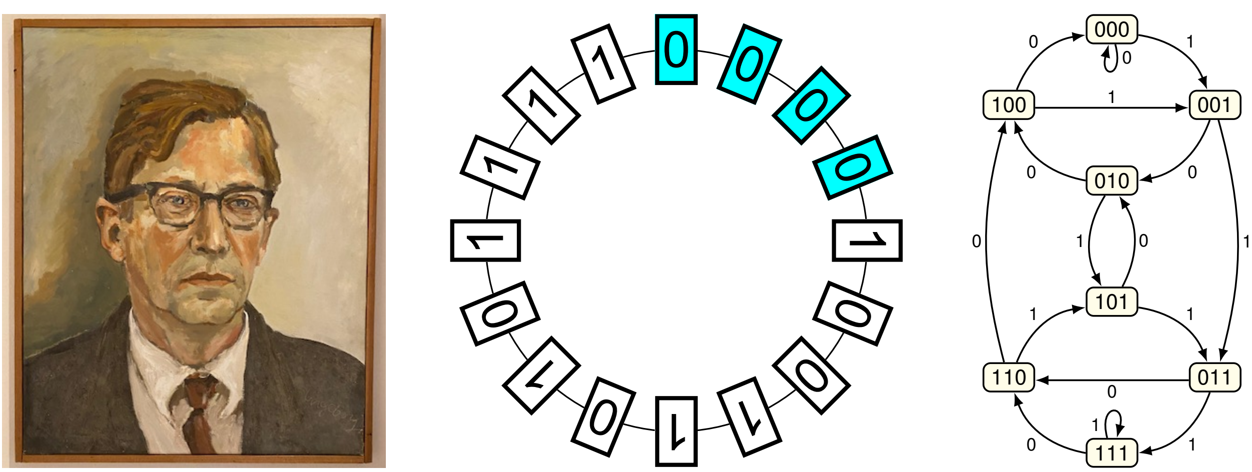

Figure 1: *Painting of de Bruijn (left; signed Ingeborg, 1970s?,* 40×50 *cm, oil on canvas); binary de Bruijn sequence of order 4, with one window of length* 4 *highlighted (center); binary de Bruijn graph of order 3 (right).*

De Bruijn's Elegant Math

Let's first look at some of the mathematics by de Bruijn that I find particularly elegant, and that has connections to the arts.

Figure 1 (center) shows a binary de Bruijn sequence over $\{0, 1\}$ of order 4. In general, a k-ary *de Bruijn sequence* of order-n over alphabet A of size k is a cyclic arrangement of k^n symbols from A such that all contiguous subsequences ("windows") of length n are distinct. Thus, each of the k^n combinations of n symbols from A occurs exactly once in a window. In 1946 [2], de Bruijn published his proof for the number of de Bruijn sequences, making clever use of (de Bruijn) graphs (Figure 1, right). These sequences have applications in engineering, but can also be used in choreography (see below: The Water Fountain) to traverse all tuples over k objects of length n in a deterministic but pseudo-random order.

In 1964, de Bruijn published some major generalizations of Pólya's Enumeration Theorem [4]. Consider domino tiles with zero to six pips at each of the two squares. How many such dominoes are there? Not 7×7, because tiles can be rotated 180°, and dominoes with the same number of pips on both squares are symmetric. So, that number equals $(7 \times 7 - 7)/2 = 21$. Pólya's Enumeration Theorem gives a recipe for constructing a formula to calculate the number of "colored" objects modulo symmetry. How many domino tiles are there with 0 through N pips? Pólya's theorem yields the formula $\left(N^2 + N\right)/2$, obtained from the structure of the domino's symmetry group C_2. And what about coloring the faces of a cube with N colors modulo its 24 rotational symmetries? That number is given by the formula $\left(N^6 + 3N^4 + 12N^3 + 8N^2\right)/24$. One generalization by de Bruijn concerns symmetries in the colors. When coloring cube faces with two colors, there are $\left(2^6 + 3 \cdot 2^4 + 12 \cdot 2^3 + 8 \cdot 2^2\right)/24 = 10$ coloring patterns. What if we consider the two colors equivalent? For instance, one blue face and five red faces is considered the same pattern as one red face and five blue faces. Using the object's symmetry group G and the symmetry group H for the colors, de Bruijn's generalization

$$\text{number of patterns} \quad = \quad P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \ldots\right) P_H\left(e^{\sum_k z_k}, e^{2\sum_k z_{2k}}, e^{3\sum_k z_{3k}}, \ldots\right)\Bigg|_{\{z_i\}=0}$$

yields 6 coloring patterns. Symmetry plays an important role in many of the arts, and counting the number of possibilities modulo symmetry helps ensure that no possibility is overlooked or duplicated.

When de Bruijn's son Frans (at age seven) could not fill a $6 \times 6 \times 6$ box with 27 bricks of size $1 \times 2 \times 4$, this led de Bruijn to formulate his theorems on filling boxes with bricks [5]. What is nowadays known as *de Bruijn's Theorem* states that a *harmonic* brick (whose side lengths a_i form a chain of divisors: a_i divides a_{i+1}) can fill a box if and only if the box is a *multiple* of the brick (that is, the box can be filled trivially by placing all bricks in the same orientation). The $1 \times 2 \times 4$ brick is harmonic, but the $6 \times 6 \times 6$ box is not a multiple of this brick (since 4 does not divide 6). De Bruijn's algebraic proof is surprisingly short, using sums of complex exponentials. This theorem has applications in puzzle design and tessellations.

Finally, I would like to mention de Bruijn's algebraic approach to aperiodic tilings of the infinite plane [7]. He considered this his finest mathematical result. We will encounter two such tilings below (The Wieringa Roof and The "Knights" Tiling). In de Bruijn's first approach, he introduces the *pentagrid*, five collections of parallel equidistant lines, where the angles between collections are multiples of 72°. There is a direct correspondence between tilings with thick and thin Penrose rhombs and de Bruijn's pentagrids. Each tile corresponds to the intersection of two grid lines. In his second approach, he related such tilings to the *projection* of the intersection of the five-dimensional cubic lattice and a planar slice. De Bruijn's methods transformed the then-nascent field of aperiodic order. Aperiodic tilings have applications in science (cf. quasicrystals). They are also artistically appealing, because they lack periodicity (which would make them somewhat dull) but are not completely random (they are quasi-periodic).

De Bruijn on Culture and Art

De Bruijn expressed his opinion about art and its relation to mathematics in various places. In his inaugural lecture at TU Delft (then known as TH Delft) [1], he chose as his main theme *mathematics as a cultural factor*. He reminded the audience that the word "culture" in the context of science used to provoke strong reactions, but that nowadays (i.e., in 1946) there were signs that this attitude had changed. And therefore he dares to relate mathematics and culture. He does not wish to propose that mathematics is a panacea for the cultural challenges of his time, but mathematics being responsible for many of the modern conveniences in the world is a cultural factor of importance. De Bruijn mentions three other such factors: mathematics as a source of indisputable knowledge, mathematics containing clear elements of beauty, and mathematics as a tool for educating people in effective thinking. As elements of beauty, he then elaborates on the *power* of mathematics (to achieve so much with so little, separate from its practical utility), the preference for *simplicity*, *elegance*, *harmony*, and *order*. He then asks whether this beauty is related to how mathematics is practiced or whether it is an inherent property of the mathematics itself. He compares this to the question whether a painting of a flower is beautiful because of the paint or because of the flower being painted. Finally, de Bruijn indicates that mathematics can offer various pleasures, such as playfulness, the excitement of exploring new (virtual) worlds, the delight of solving a problem, and the joy of surprising eureka insights.

For the Escher exhibition at the ICM in 1954, he wrote the preface of the accompanying catalog [3]:

> "In view of the fact that Mr. Escher's work may be said to be a point of contact between art and mathematics, the Organizing Committee of the International Congress of Mathematicians 1954 (Sept. 2nd–9th) at Amsterdam, took the initiative in inaugurating this exhibition.
>
> Probably mathematicians will not only be interested in the geometrical motifs; the same playfulness which constantly appears in mathematics in general and which, to a great many mathematicians is the peculiar charm of their subject, will be a more important element.
>
> It will give the members of the Congress a great deal of pleasure to recognize their own ideas, interpreted by quite different means than those they are accustomed to using."

In his farewell lecture upon retiring from TU Eindhoven [8], he said (translated from Dutch by the author):

> "It seems that one can admire the result of painting without placing oneself in the role of the creator of the artwork. In music, an intermediate appreciation is sometimes possible, as the listener may still attempt to identify with the performing artist but not with the composer. In mathematics, it seems more often the rule that if one cannot, in any way, get the impression of ever having been able to create a subject oneself, and if one cannot play any creative role in it, then one cannot grasp it—and cannot admire it either. In short, to appreciate mathematics is to practice mathematics."

The Royal Dutch Academy of Arts and Sciences (KNAW), of which de Bruijn was a member, invited him to participate in a dialog on science and art to celebrate the 190-year anniversary of the academy. De Bruijn carefully wrote out his speech [10], in which he focused on mathematics and art, pointing out many similarities and differences. (It is my intention to publish a translation of this speech separately.)

In a later interview about Escher [11], de Bruijn stated:

> "if it comes to beauty I am not really such an admirer of Escher. [...] I understand how it was created, but Escher's artistic interpretation[1] was somewhat wooden, out of sheer necessity. His human figures looked like wooden puppets. That is why regular artists did not want to call it art; what he did was not something they could recognize."
>
> "for mathematicians, Escher's work is definitely entertaining, but beautiful... What is beauty? I really do not know."

[1] Escher himself admitted that he absolutely could not draw.

Dick de Bruijn as Creator and Performer

Besides having contributed creative mathematics in diverse fields, de Bruijn also was creative in other ways. He played chess and it is said that he created chess problems, which he discussed with Max Euwe (a Dutch mathematician and chess player; World Champion Chess 1935–1937). He was instrumental in establishing the Dutch National Mathematical Olympiad in 1962. Some of his math problems featured on the International Mathematical Olympiad (IMO), notably Problem 6 at IMO 1993 and Problem 2 at IMO 2005. De Bruijn was to a large extent an autodidact, which, I think, motivated him to write articles for *Pythagoras*, the math journal for Dutch high-school students (started in 1961 and still in existence), and for *Euclides*, the journal for Dutch math teachers. He also designed some puzzles and liked conjuring and card tricks [9].

Dick loved writing poems, in particular limericks, and he was quite good at that. He produced them by the dozen. Of the forty or so limericks or limerick-like poems (all in Dutch) that I have seen, only one comes close to the field of mathematics. It involves a Dutch neologism: in the verb “programmeren” (Eng.: to program), he replaces “meren”, which relates to “meer” (Eng.: more) by “minderen”, which mean “to lessen” (to the right is my literal translation):

Een Wijze sprak ernstig: Weet, kinderen	A Sage spoke earnestly: Know, children
dat domme studenten mij hinderen	that stupid students hinder me
Perfect programmeren	Perfect programming
valt hun niet te leren	they cannot be taught
’t Is beter dat zij programminderen	It is better that they program⌒less

When a student in Leiden (1936–1941), Dick used to invite a group of math friends to his home on Friday evenings. They played Bridge, debated, lectured to each other, and he played Jazz records (in particular, Duke Ellington) on the electric gramophone of his brother Johan. At two social activities for students, they performed as vocal group under the name *The Epsilon Boys*. Dick had written the song texts, something he enjoyed (viewing it as problem solving, I imagine). One of the songs was about the Dutch mathematician H.D. Kloosterman. Later, he also sang at several occasions, including the Math Department party in 1979, for which he had written a song about Mrs. Geerts who ran the department’s canteen for many years.

Dick played the piano (some say quite well, though he could not read musical scores). He mostly played (pre-1960) Jazz and liked to improvise. For a while he tried the saxophone. Another love was drawing, for which he organized weekly sessions with some friends, until late in his life. He had previously taken lessons from Hugo Brouwer, a well-known Dutch painter, mosaic and glass artist, and sculptor. Dick also did some painting and sculpting, though almost exclusively non-mathematical in nature.

Artwork by de Bruijn

De Bruijn designed only a few physical artifacts that can be considered (mathematical) art. Two of these still exist, one was disassembled, and another one is still somewhat of a mystery. These are discussed in the following subsections.

The Blue Knot

While playing with a plastic construction kit for children, known as *Plasticant*, de Bruijn wondered whether it would be possible to construct a cloverleaf knot that has only right angles and that has dihedral symmetry (order 2 and order 3 rotations). He describes his design consisting of nine segments in [6]. Since Plasticant is blue, he called this *The Blue Knot*, which is also the nickname for the Dutch association of teetotalers. A larger PVC version (Fig. 2, left) hung in the Math Library in the Main Building at TU Eindhoven until the department moved to another building in 2012. I was able to recover it from a messy storage room.

Figure 2: *Blue Knot by de Bruijn (left; PVC, 70 cm diameter); Dick in the Math Colloquium Room with The Wieringa Roof on the back wall (image credit: Collection of Peter van Emde Boas, 1987).*

The Wieringa Roof

Shortly after de Bruijn published his pentagrid and cut-and-project methods for constructing aperiodic Penrose tilings [7], he collaborated with the Department of Mechanical Engineering on such a tiling realized with aluminum rhombs. His student Rob Wieringa had noted that the vertices of a thick and thin rhombus tiling could be raised out of the plane such that all rhombs became golden rhombs. Depending on their elevation angle they would project back into thick and thin Penrose rhombs. Since such a tiling is no longer flat, it would not be suitable for floors, and de Bruijn dubbed it *The Wieringa Roof*, which hung in the Math Colloquium Room in the Main Building at TU Eindhoven until 2012 (Fig. 2, right). Actually, Rob was the one who had selected the final design and put it together.

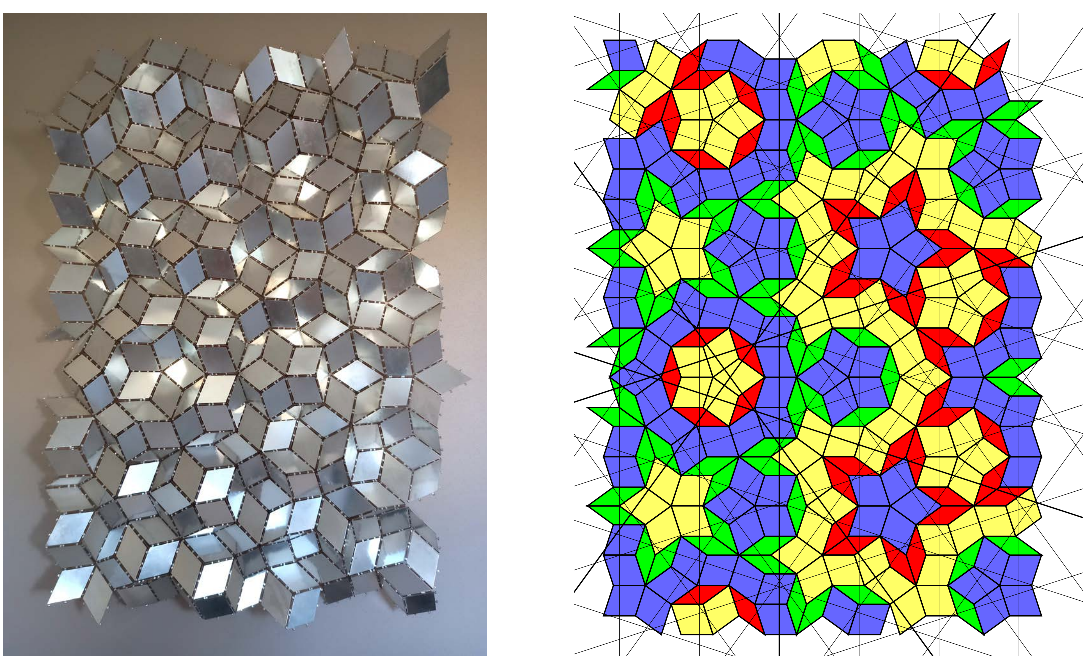

Figure 3: *Restored Wieringa Roof by N.G. de Bruijn and R.M.A. Wieringa (left; aluminum,* 120×90 *cm); height-colored reconstruction with grid lines, extended to the right to spot some mistakes (right).*

Unfortunately, it had ended up in that same messy storage room. In this case, Rob Wieringa managed to get hold of it and restore it (Fig. 3, left). Since there are infinitely many different thick+thin rhombus tilings, I was interested in finding out what parameters were used for *The Wieringa Roof* (in fact, de Bruijn proved that a pentagrid is determined by a single complex number ξ). This is actually a hopeless task, because any finite fragment of such a tiling appears infinitely often in any other tiling. In fact, by a theorem of John Conway, if the diameter of the fragment is d, then you will find that fragment in any disk with diameter πd. However, Rob Wieringa remembered that *The Wieringa Roof* has a pseudo-reflection line along a *Conway worm*. And I noticed that perpendicular to this line, there is a real reflection line. There exist only three pentagrids with these symmetries. *The Wieringa Roof* turns out to agree with $\xi = 5/2$. The reconstruction from this parameter is shown in Fig. 3. It revealed that there are a few mistakes in the original roof (look carefully along the right-hand edge).

The "Knights" Tiling

At his home, de Bruijn had a tiling he was very proud of. I have only found a vague picture of it (Fig. 4, left). Together with Walt van Ballegooijen, we were able to come up with a reconstruction (Fig. 4, right). As you

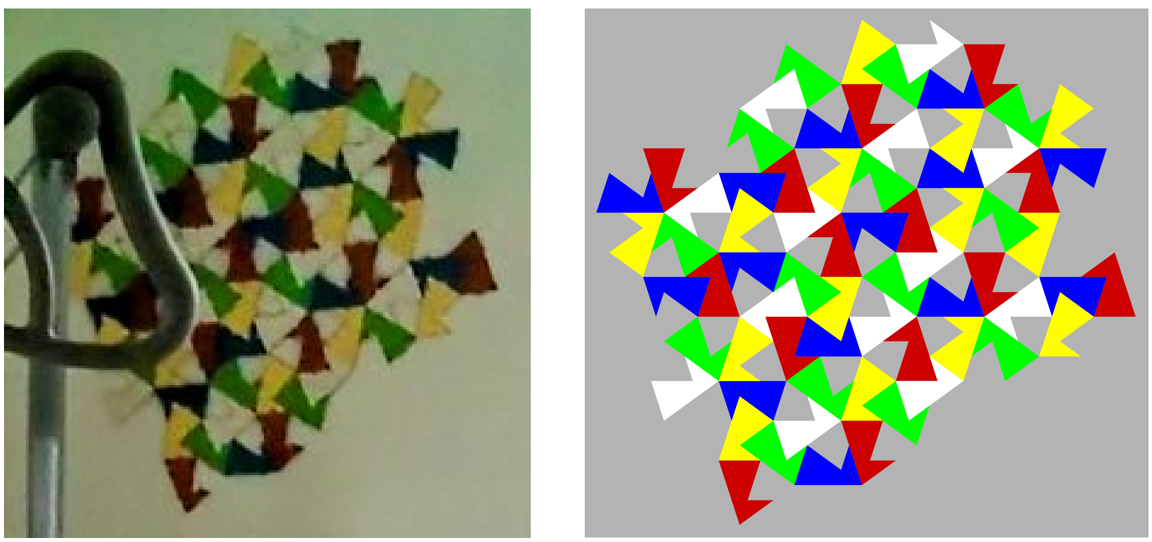

Figure 4: *The "Knights" Tiling by N.G. de Bruijn (left; plastic tiles,* 50×50 *cm) and reconstruction (right).*

can see, it is a tiling with tiles that look like a knight chess piece and holes in the shape of a kite. Knight colors vary by orientation. It turns out that the knight can be split into a kite and a dart (Fig. 5). When converting to thick+thin rhombs (Fig. 6), it appears that also this tiling has two perpendicular reflection lines (one of which is a pseudo-reflection along a vertical Conway worm, in cyan). In this case, we have $\xi = \frac{1}{2}\left(e^{2\pi i/5} - e^{-2\pi i/5}\right)$. It turns out that one of the knights is in an illegal position: the white knight at the bottom left.

The Water Fountain

Jan Donkers and Frans de Bruijn told me that Dick had designed a water fountain, which was actually constructed somewhere outside on the TU campus (sometime between 1965 and 1975). Henk van Tilborg mentions a computer that worked on water to be installed indoors, but that was never realized (this may have been in the mid 1980s). These waterworks remain a mystery. I still need to consult de Bruijn's archive, which is stored in Haarlem. In the meantime, I fantasize about how de Bruijn could have choreographed the spouts of a water fountain. I myself would probably have used a *Gray code*, which is a way of presenting all N-bit patterns changing only one bit at a time, to control the on/off state of N spouts. But it would have been more in line with de Bruijn's research to use a binary de Bruijn sequence of order N, which also looks nicer.

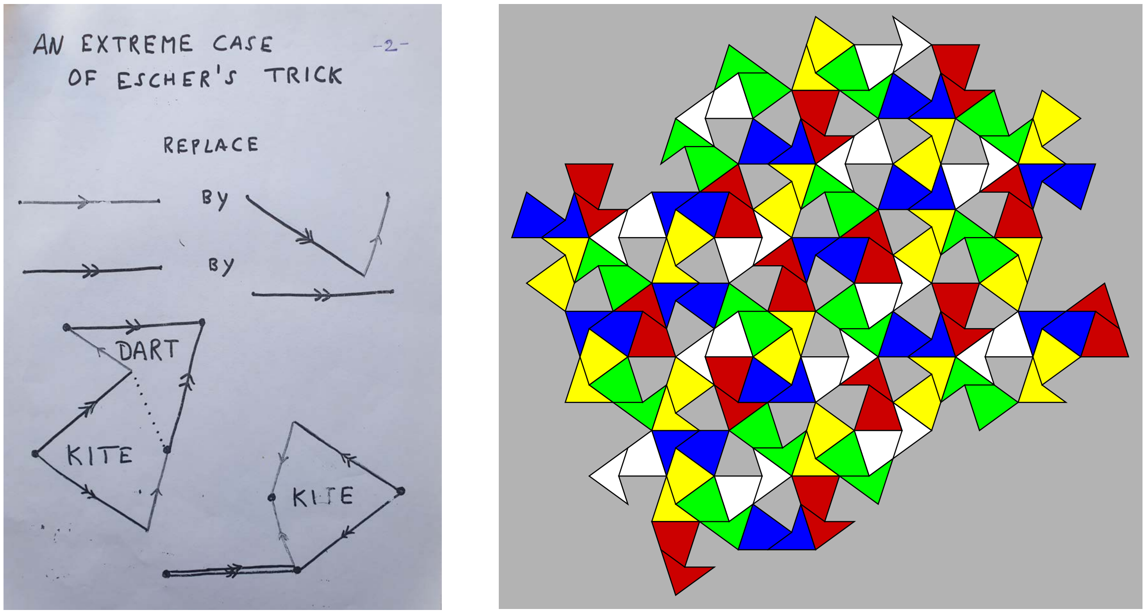

Figure 5: *Slide by de Bruijn with "knight" shape consisting of a kite and a dart (left); corrected "Knights" Tiling showing the underlying kites and darts (right).*

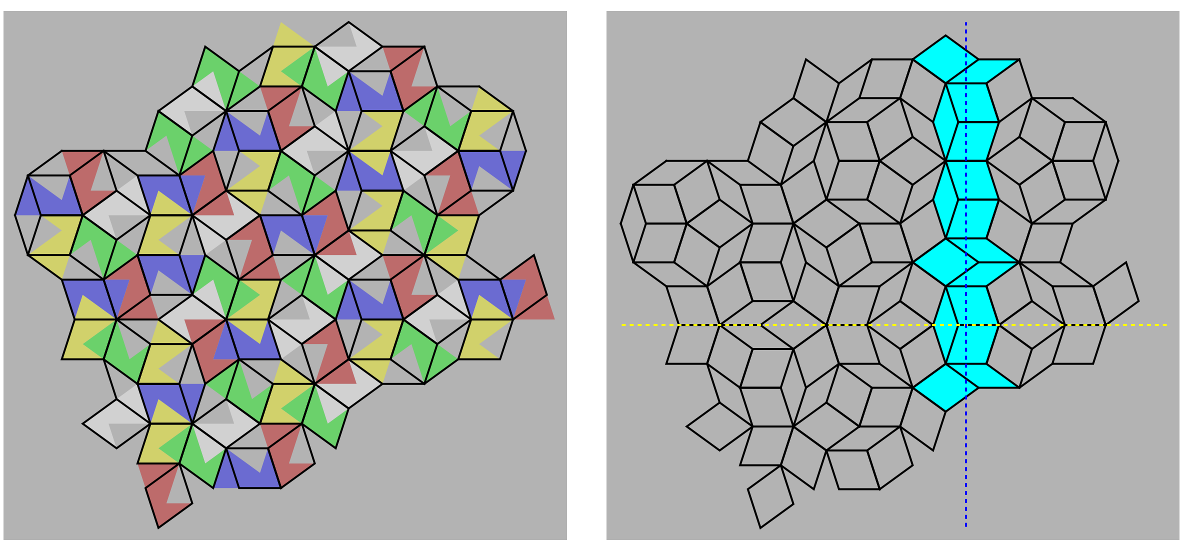

Figure 6: *The "Knights" Tiling with thick and thin rhombs superposed (left); thick and thin rhombs with symmetry lines and Conway worm (right).*

Conclusion

I have given you an impression of the many ways in which Dutch mathematician Dick de Bruijn connected mathematics and the arts. As a former student and colleague of Dick at TU Eindhoven (1976–2012), I have been close to the action, so to say, but I never collaborated directly with Dick. In the online supplementary material, I provide further details.

Acknowledgments

I would like to thank the many people that I have spoken while researching the connections between de Bruijn, mathematics, and art. In particular, I would like to mention his youngest daughter Judith and his son Frans, who provided relevant pictures and stories. I would also like to thank Doris Schattschneider and Marjorie Senechal for suggesting this topic and for helping me improve the paper.

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