Geometrically Regular Handle-Bodies

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Abstract

This is an attempt to generalize the concept of the *geometrically regular* shapes of the Platonic polyhedra to objects with curved edges and faces that are all identical, and where faces are not restricted to be topological disks. Some of the objects studied, are "globes" of genus zero, bi-pyramids, and Platonic edge-frames with twisted branches, as well as mathematical knots embedded as edges on the surface of handle-bodies of higher genus.

Introduction

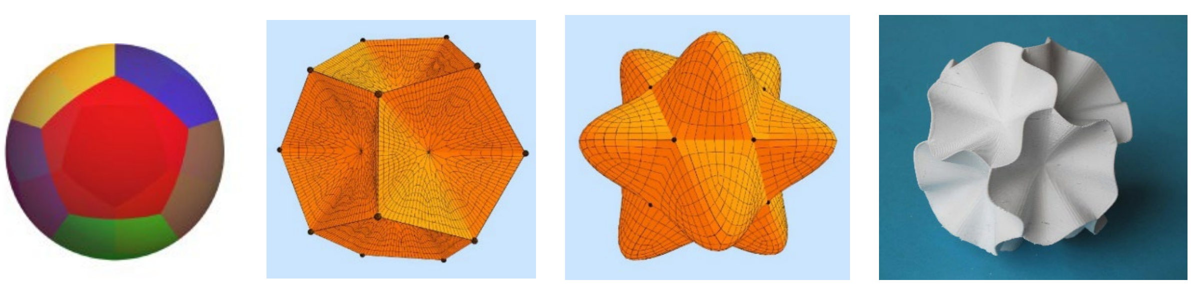
In the five *Platonic polyhedra*, all vertices, (straight) edges, and (planar) faces are exactly the same. Moreover, all *flags* (combinations of a vertex and its adjoining edges and faces) can be transitively transformed into one another using a symmetry of the polyhedron. A generalization of those *geometrically regular shapes* allows edges to be curved, and faces to be non-planar.

In topological graph theory, a *regular map* is a conceptual decomposition of a *two-dimensional manifold* (such as a sphere, torus, or Klein bottle) into *topological disks*, so that every flag can be transformed into any other flag by a topology-preserving operation. For most regular maps [2] no *geometrically regular* realization is possible in 3D Euclidean space.

In this paper, I am studying the tessellation of *orientable handle-bodies* into more general *regular geometrical shapes* that have all identical flags, and their "faces" or "countries" are not limited to topological disks; they may be of higher genus (i.e., disks with holes). I introduce my exploration with a generalization of the Platonic polyhedra to *geometrically regular* shapes of genus zero with curved features. Then I expand the scope to shapes of higher genus, like Tord Tengstrand's intriguing "3-2-1"-sculpture [6], presented at Bridges 2020, and to some derivative bi-pyramidal designs [4]. I continue with regular tessellations on symmetrical handle-bodies that are derived from the edge-structures of the Platonic solids. I also explore shapes derived from symmetrical configurations of mathematical knots. The search for other families of *geometrically regular* shapes continues.

Geometrically Regular Globes

A simple approach to construct "Platonic" objects with curved features, is to project the Platonic polyhedra onto their circum-spheres, as is shown in Figure 1(a) for the pentagonal dodecahedron. In this transformation, the edges become simple circular arcs, and the faces become spherical patches. Clearly, the congruence between the vertices, edges, and faces is maintained.



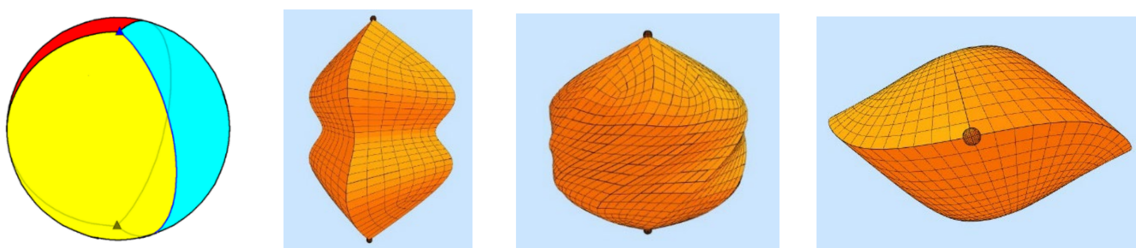
(a) (b) (c) (d)

Figure 1: *Platonic dodecahedral balloon globes: (a) spherical shape; (b) straight edges, concave faces; (c) straight edges, convex faces; (d) 3D-print with curved edges, undulating faces.*

But there are many more degrees of freedom to make shape changes that still maintain the "Platonic" topology. Another simple transformation maintains the straight edges but makes the polyhedron surface from some stretchable material, like a rubber balloon. If the air pressure inside the balloon is lower than the pressure on the outside, all faces will warp into concave areas with positive Gaussian curvature (Fig.1b). If the outside pressure is lower than the inside pressure, the faces will form outwards bulges, also with positive curvature (Fig.1c).

To obtain even fancier shapes, the edges can be given an "S"-shape with rotational C_2-symmetry around their midpoints (Fig.1d). With the surface material mimicking the properties of a soap film and even pressures on the inside and outside of the Platonic ball, all the faces would become minimal surfaces with negative Gaussian curvature. All of these shapes maintain the symmetry of the oriented dodecahedron.

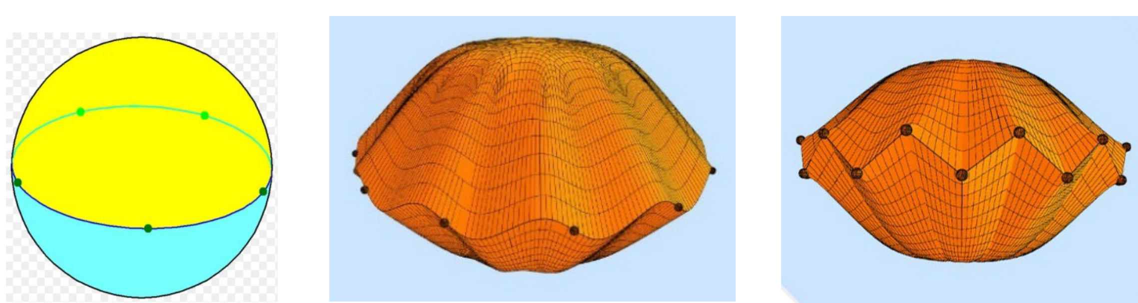
Figure 1 inspired me to construct *geometrically regular* shapes of genus zero that have a number of vertices that does not appear in the Platonic polyhedra. With only two vertices, we can form *hosohedra* with any number of slices for all positive integer numbers. The classical trigonal hosohedron is shown in Figure 2(a). But now we can give the edges running from the North pole to the South pole more interesting shapes than simple circular arcs. In Figure 2(b) the edges have a planar, undulating shape. In Figure 2(c) I have twisted a 9-sided hosohedron, so that its edges take on spiraling, helical shapes. We can even draw a map with just one vertex on a single C_2-symmetrical, closed-loop edge, which partitions the sphere into two identical halves (Fig.2d). All of these constructions are geometrically regular.



(a) (b (c) (d)

Figure 2: *(a) Trigonal hosohedron; (b) 3-sided hosohedron with planar, non-circular edges; (c) twisted 9-sided hosohedron; (d) a regular shape with a single vertex.*

The duals of the hosohedra are the *dihedra*. Figure 3(a) shows a pentagonal dihedron. This structure now allows us to place any number of vertices along an equatorial circle and modify the edge segments between neighbors, – for instance into an undulating "S"-shape (Fig 3b). We can even alternatingly offset the vertices from the equatorial plane, while keeping all edges straight (Fig.3c), or giving them C_2-symmetry about the edge midpoint.



(a) (b) (c)

Figure 3: *(a) Pentagonal dihedron; (b) "wavy" 12-vertex dihedron; (c) 24-vertex dihedron with vertices that are vertically offset from the equatorial plane.*

This is already a surprisingly rich set of possible *regular* "Platonic" surfaces. But the exploration gets even more interesting, when we start looking at handle-bodies of higher genus.

Bi-Pyramid Structures: Tord Tengstrand's "3-2-1"-Sculpture and Derivatives

This exploration was inspired by an intriguing sculpture (Fig.4a) titled "3-2-1", presented by Tord Tengstrand [6] in the Bridges 2020 Art Exhibition. It has three curved edges, two "pointy" vertices, but only one complicated curved "face," which is topologically equivalent to a disk with two holes. This surface is formed by three "ribbon-countries" lying between the three sharp edges that have a dihedral angle of about 60 degrees (shown in red, green, and blue) (Fig.4b). The edges each connect the two vertices by a vertical "down-up-down" zig-zag move, while passing the equatorial plane between the two vertices three times. I call this a *3-pass edge*. The ribbon countries have a similar 3-segment zig-zag shape, and their ends join in the two 3-way junction areas into a single smoothly-connected face. The result is the orientable surface of a solid handle-body of genus 2, corresponding to a 2-hole torus or, equivalently, to the thick shell of a hollow sphere with three "tunnels" or "windows" to the interior void. This genus 2 handle-body has 3-fold rotational D_3-symmetry around an axis that passes through the two (black) vertices in Figure 4(b).

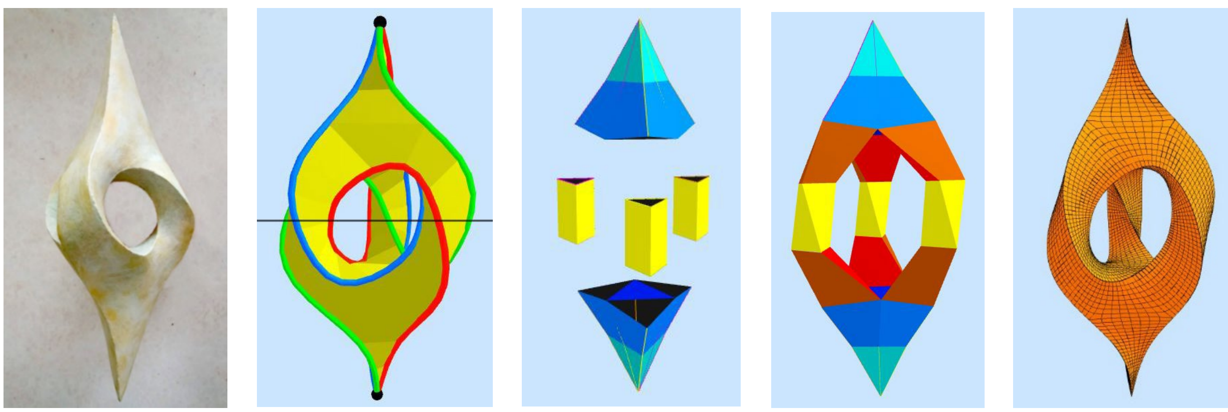


Figure 4: *(a) Tengstrand's "3-2-1"-sculpture; (b) CAD model showing the 2 vertices and the 3 edges; (c) modular starting geometry; (d) polyhedral shape; (e) smoothed by Catmull-Clark subdivision.*

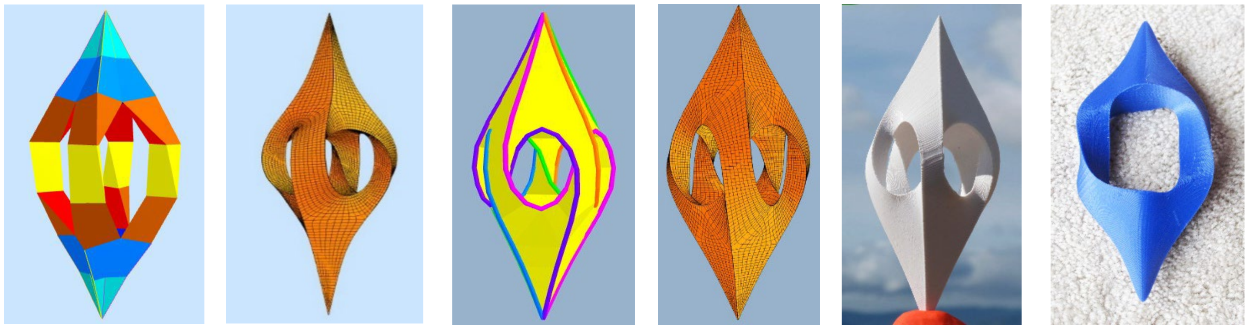
A practical way to model the target shape (Fig.4a) is to form the three solid branches with 3-sided prismatic (yellow) beams (Fig.4c), which are then twisted and connected to the two 3-sided (blue) pyramids that support the top and bottom vertices. This results in a polyhedral model (Fig.4d). The facets that form the ribbon-countries are combined, and the edges between them are labeled as "sharp." The model is then subjected to three levels of Catmull-Clark subdivision [1] in which the designated "sharp" edges maintain their dihedral sharpness. This results in nice, smoothly-shaped face strips (Fig.4e), even when the starting polyhedral model is rather coarse.

Changing the Genus by Constructing "N-2-1"- Bi-Pyramids

Tord Tengstrand's sculpture inspired me to design other derivative shapes with interlinked curved edges and a complex symmetrical surface with a genus greater than two. In my first effort to extend the Tengstrand family, I simply increased the rotational symmetry of the original bi-pyramidal structure [4].

Figures 5(a,b) show the construction of a "4-2-1"-handle-body of genus 3, with overall D_4-symmetry. Figures 5(c,d,e) show models of D_5-designs. Figure 5(c) shows the five interlinked loopy edges, and Figure 5(e) is a model made on a low-end 3D-printer [7] by converting the surface geometry to an STL-file. In all these bi-pyramid structures, the edges still make three passes through the central equatorial plane when going from one vertex to the other one.

A similar approach also allowed me to make a bi-pyramid with only two branches (Fig.5f).



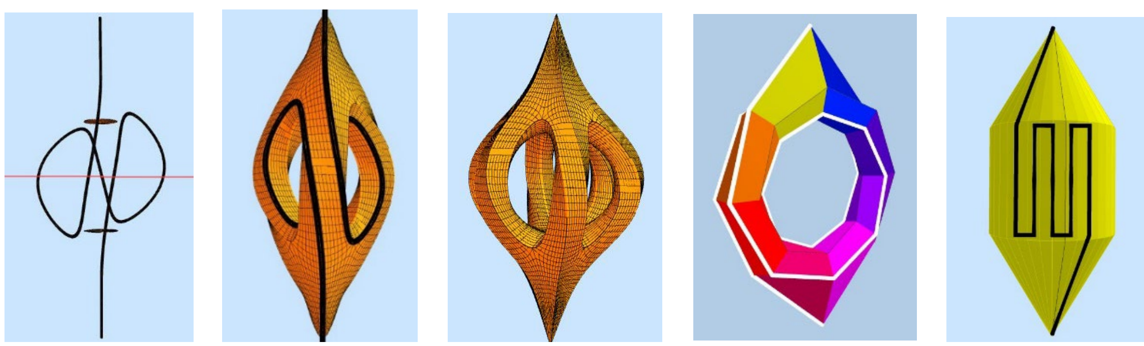
(a) (b) (c) (d) (e) (f)

Figure 5: *Bi-pyramidal structures with 3-pass edges: (a, b) 4-branch "4-2-1"-models of genus 3; (c ,d, e) 5-branch models and sculpture; (f) 2-branch "2-2-1"-bi-pyramid sculpture of genus 1.*

More Complicated, Multi-Pass Edges

Another way to extend the bi-pyramid family is to use more complicated edge curves, e.g., edges that pass the central void five times on their way from one vertex to the other one (Fig.6a). These 5-pass edges can be used on bi-pyramids with any number of branches. To accommodate all the extra curve segments, all branches of the bi-pyramid now have a pentagonal cross-section (Figs.6b,c). We can also use a 5-pass edge on a 2-branch structure (Fig.6d); the edge now travels five times through each of the two branches.

As an extreme case, we might ask, what would happen if we had just a single 5-sided prismatic branch connecting the two pyramid tips. Topologically this becomes equivalent to a hosohedron with a single edge connecting the two poles: but this edge now zig-zags up/down past the middle five times (Fig.6e).



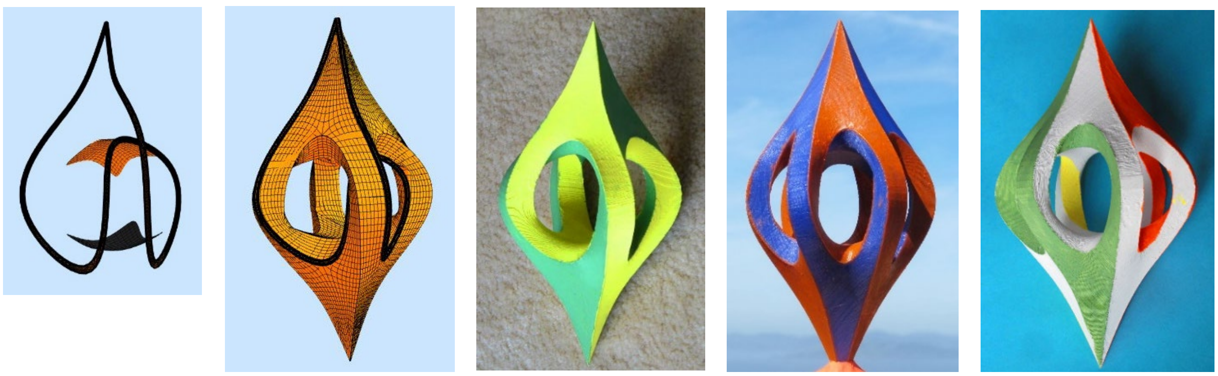
(a) (b) (c) (d) (e)

Figure 6: *5-pass edges between the two pyramid tips: (a) one edge; (b) 3-branch "3-2-1"-model; (c) 6-branch "6-2-1"-model; (d) 2-branch toroid; (e) 5-pass edge on a single-slice hosohedron.*

All of these shapes are *geometrically regular*; their edges and vertices are all the same, and there is just a single smoothly-connected "face" that has the appropriate symmetry corresponding to the number of edges and the valence of the two vertices. Of course, when there is just a single edge, the concept of geometrical regularity becomes rather "anemic." This single edge must have C_2-symmetry around its mid-point, so that the two vertices "are the same" and can be transformed into one another with a 180° rotation.

When we use 4-pass edges (Figs.7a-e), or, more generally, edges with an even number of passes, a complication arises. Now each edge ends on the same vertex that it started from, and the bi-pyramid needs to have an even number of branches. This reduces the rotational symmetry by a factor of two. Moreover, there is an interesting effect: The extra central bend in the edge curve (Figs.7a,b) cuts off the access of one of the ribbon countries to one of the two shared junction areas located at the bases of the two pyramids. Still, the four initial ribbon countries between the four edge curves can meet in pairs at the two junction areas, resulting in two smoothly-connected Tengstrand "faces," shown in yellow and green in Figure 7(c).

Similarly, 4-pass edges on a 6-branch bi-pyramid also divide the surface area into separate "faces." If the loopy edges starting at the top and bottom vertices are properly oriented against one another, three of the initial ribbon countries merge in the top junction area (blue), and the other three join in the bottom (red) junction, resulting in just two final "faces" (Fig.7d). However, if one set of edges is rotated by 120°, one set of three ribbon countries (white) merge redundantly in both junctions, thus preventing the other three countries from accessing any junction area; they thus remain isolated, and there are then four separate, smoothly-connected "faces" (Fig.7e). But they are not all the same, and this is no longer a *regular* object.

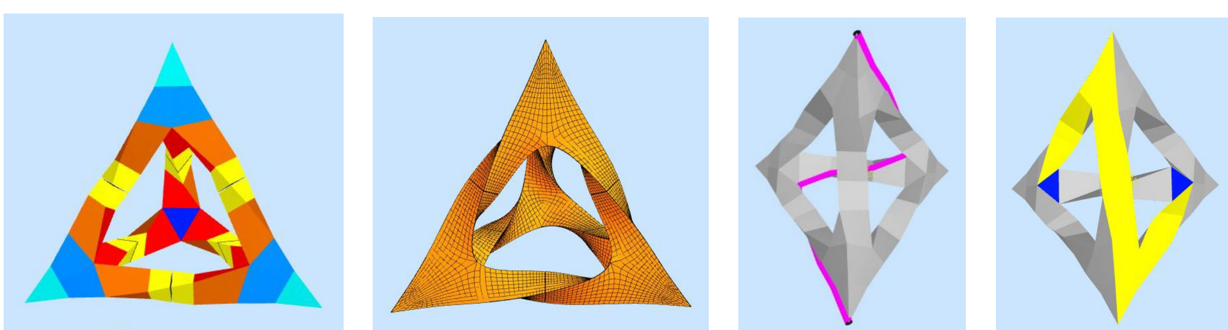


(a) (b) (c) (d) (e)

Figure 7: *4-pass edges: (a) One 4-pass edge; (b) 4-branch model; (c) 4-branch "4-2-2"-sculpture; (d) regular 6-branch "6-2-2"-shape; (e) non-regular "6-2-4"-shape.*

Platonic Edge-Frames with Twisted Branches

If I split Tengstrand's original "3-2-1"-sculpture along the equatorial plane into two equal halves, I obtain two pyramids with three prismatic "legs." By adjusting the angles between those legs, I can create a modular corner unit, where four of them can combine into a tetrahedral frame (Fig.8a). Maintaining the original twist in the legs, I obtain an interesting object (Fig.8b). It has six 3-segment (purple) edges (Fig.8c) between pairs of vertices and also six 3-segment (yellow) ribbon countries (Fig.8d) between two (blue) 3-way junction areas. All the ribbon countries connect through the four junction areas into a single, multi-branch, smoothly-connected Tengstrand "face." In this handle-body it was possible to make all the edges and vertices congruent to one another; so this is also a *geometrically regular* object, and in Tengstrand's "E-V-F"-notation this is a "6-4-1"-structure.



(a) (b) (c) (d)

Figure 8: *Tetrahedral edge-frame: (a) Four 3-leg pyramids; (b) smoothed frame; (c) one 3-segment (purple) edge between two vertices; (d) one 3-segment ribbon country between two (blue) junctions.*

This same treatment can be applied to all five symmetrical frames based on the edge structures of the Platonic solids. The resulting number of "faces" may differ for different numbers of segments in the edges (or ribbon-countries). Figures 9 and 10 show some examples. If I give the tetrahedral frame pentagonal prism branches to accommodate 5-segment edge, all ribbon countries still merge with all the other ribbon countries into a single Tengstrand face, resulting in another "6-4-1"-structure (Fig.9a). When the cube-frames are covered with 3- or 5-segment ribbon countries, three ribbon-countries merge in a pair of two opposite 3-way junction areas, resulting in "12-8-4"-structures (Figs.9b,c). Similarly, in a 3-segment octahedral frame, four 3-segment ribbon-countries merge in two opposite 4-way junction areas, resulting in a "12-6-3"-structure (Fig.9d).

(a) (b) (c) (d)

Figure 9: *(a) 5-segment tetrahedral frame; (b) 3-segment cube-frame; (c) 5-segment cube-frame; (d) 3-segment octahedral frame forming a "12-6-3"-structure.*

However, when we use pentagonal branches and 5-segment edges on the octahedral frame, all ribbon-countries merge into one single "face" (Fig.10a). In 3-segment or 5-segment dodecahedral frames all ribbon-countries also join into just one single face, forming a "30-20-1"-Tengstrand-structure (Fig.10b). Similarly, for icosahedral frames, both the 3-segment version (Fig.10c), as well as a 5-segment version, have just a single face, and they form "30-12-1"-structures.

(a) (b) (c) (d)

Figure 10: *(a) 5-segment octahedral frame; (b) 5-segment dodecahedral frame; (c) 3-segment icosahedral frame; (d) 3-segment cuboctahedral "24-12-4"-frame.*

More Twisted Frame Structures – Ongoing Work

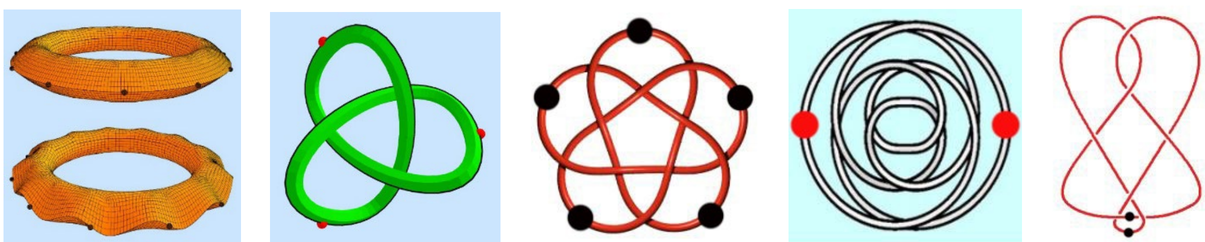
The exploration of twisted frames is far from complete. Some Archimedean solids also have all identical edes that may lead to the construction of geometrically regular frames. Figure 10(d) shows a 3-segment cuboctahedral "24-12-4"-structure, with the four merged super-countries shown in different colors.

Several other issues need to be studied further. Will the 4-pass edges, based on the 4-leg corner module used in Figure 7(c), cause extra problems as they did for the 4-branch bi-pyramids, or will even-numbered segments lead to additional interesting structures with more than just one face? In all the frame structures studied so far, the twist of the individual branches has been kept to a minimum. Will increased twistiness lead to conceptually new structures? Are there other approaches that lead to new families of geometrically regular shapes? It seems plausible that all of them need to start from a handle-body with all identical handles. Would it be possible to embed a geometrically regular edge-pattern on a frame that replaces every edge of a Platonic solid with two side-by-side branches?

Toroidal Handle-Bodies of Genus One

Another approach to constructing geometrically regular handle-bodies starts with tubular shapes following toroidal rings or mathematical knots. On a simple torus, one can run a single circular edge connecting the outermost points in the equatorial plane. Inspired by the *n*-sided dihedra (Fig.3), we can then place gratuitously many vertices on this edge and obtain a regular shape with just a single face; and we even have the freedom to modify the edge segments between neighboring vertices in a consistent manner (Fig.11a), maintaining a geometrically regular “*n*-*n*-1”-structure.

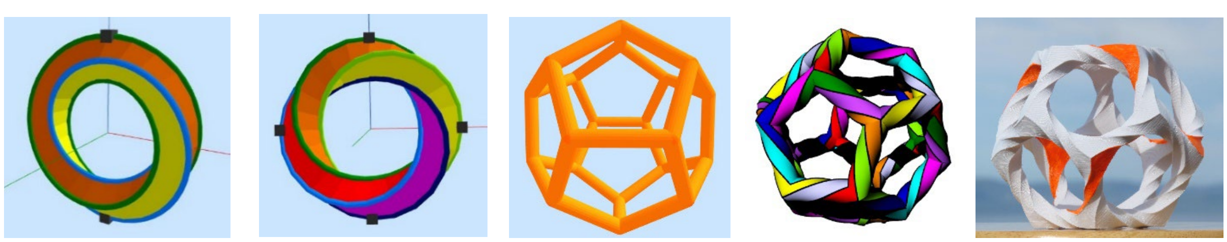
If the toroid has less rotational symmetry, e.g., 3-fold as in the trefoil knot (Fig.11b), then this defines the number of vertices that must be placed in corresponding locations on the edge loop. This allows us to take a symmetrical layout of a mathematical knot and turn it into a geometrically regular object. The pentafoil (Fig.11c) then becomes a “5-5-1”-structure. Knot_6_3 (Fig11.d) also can give rise to a geometrically regular “2-2-1”-structure. Even knot_7_6 can be drawn with 2-fold symmetry (Fig11.e). But the two points where the knot curve crosses the 2-fold symmetry axis of the layout are not congruent. Placing just a single vertex at one of these two points leads to a “1-1-1”-structure, in which every vertex, edge, or face maps onto itself. However, this renders the concept of “*geometrical regularity”* rather useless.



(a) (b) (c) (d) (e)

Figure 11: *(a) Toroids with 12 vertices on the outer equatorial circle; (b) single-edge trefoil knot; (c) pentafoil knot; (d), 2-fold symmetrical layout of knot_6_3; (e) a special layout of knot_7_6.*

Interesting constructions emerge when we run more than one edge along the toroidal tube, essentially forming identical, intertwined torus knots that actually partition the toroidal surface. Figure 12(a) symmetrically interlinks two (1,2)-torus knots with a phase shift of 180° between them, resulting in two interleaved ribbon-country loops and forming a “2-2-2”-structure. Figure 12(b) interlinks four (1,1)-torus knots with four interleaved ribbon-country loops between them, forming a “4-4-4”-structure.



(a) (b) (c) (d) (e)

Figure 12: *(a) Toroidal “2-2-2”-structure; (b) toroidal “4-4-4”-structure; (c) tubular dodeca-frame; (d) 10 super-countries on a tubular dodecahedron (by P. Gailiunas); (e) twisted dodecahedron frame.*

Another productive way to model regular handle-bodies starts with a symmetrical tubular frame, as shown for the dodecahedron (Fig.12c). To implement an *n*-pass edge network, we then paint *n* identical helical “edge”-curves onto each tube segment, which we then connect to one another as well as to single vertices at the tip of each frame corner. This partitions the overall tubular surface into *n*-segment ribon-countries, many of which smoothly connect to one another at the junction areas inside each corner. Paul Gailiunas has found several interesting configurations that result in more than just one single final face. Figure 12(d) shows a particularly nice “30-20-10”-structure on a dodecahedral frame, which results when each branch has an enhanced twist of 144°, rather then the minimal twist of 72°. This finding then allowed me to design a correspomding twisted polyhderal frame (Fig.12e). One of the ten super-countries has been highlighted.

Another source of inspiration for developping geometrically regular shapes are the crocheting models by Dong and Torrence. Figure 13(a) shows a trefoil knot embedded in a handle-boduy of genus 2, which was given to me by Eve Torrence. It is based on a design by Shiying Dong [3]. It can be understood as the orientable, two-sided Seifert-surface [8] of a trefoil knot (Figs.13b,c,d), where the “soap-film”-surface has been thickened to form a solid handle-body of genus 2. The trefoil is now a line embedded on this handle-body. Three vertices can be placed at the outermost points of the three lobes. This results in a geometricly regular object with a “3-3-2”-structure (Fig.13e).

(a) (b) (c) (d) (e)

Figure 13: *(a) Crocheted trefoil knot on a genus-2 handle-body; (b,c,d) Seifert surfaces on trefoils; (e) geometrically regular CAD-model with trefoil knot emphasized.*

In summary, several approaches have been presented to create a few families of shapes that are *geometrically regular*. Probably, there are other approaches that can lead to additional interesting regular shapes. Perhaps some *regular maps* [2] can be instantiated with enough symmetry, so that they can be made *geometrically* regular. But even more exciting would be to find new regular shapes where the faces are of non-zero genus. I would like to hear from interested readers, so that the exploration can be widened.

Acknowledgments

Thanks to Tord Tengstrand, whose “3-2-1”-sculpture inspired me to derive other frame structures with twisted prismatic branches. Thanks also to Paul Gailiunas and to the members of the BAAM community (Bay Area Artists and Mathematicians) who stimulated the broad exploration presented in this paper.

References

[1] E. Catmull, J. Clark. “Recursively generated B-spline surfaces on arbitrary topological meshes.” (1978) – https://people.eecs.berkeley.edu/~sequin/CS284/PAPERS/CatmullClark_SDSurf.pdf

[2] M. Conder. “Orientable regular maps of genus 2 to 101.” (2006). – https://www.math.auckland.ac.nz/~conder/RegularOrientableMaps101.txt

[3] S. Dong. “Genus-2 stuffed toy.” (2023). – https://www.youtube.com/watch?v=0BbV9tt2-hU

[4] C. H. Séquin, "Handle-Bodies Inspired by Tengstrand's "3-2-1"-Sculpture." Bridges Conference, 2024. – https://people.eecs.berkeley.edu/%7Esequin/PAPERS/2024_Bridges_Tord.pdf

[5] C. H. Séquin, "Tengstrand's "3-2-1"-Sculpture and New Derivatives." SCULPT at SMI Conference, 2024. – https://people.eecs.berkeley.edu/%7Esequin/PAPERS/2024_SCULPT_Tord.pdf

[6] T. Tengstrand “3-2-1-Sculpture.” Bridges 2020 Art Exhibit – https://gallery.bridgesmathart.org/exhibitions/2020-bridges-conference/tordtengstrandteliacom

[7] Ultimaker. “Reliable 3D printers.” – https://ultimaker.com/3d-printers

[8] J. J. van Wijk, A. M. Cohen. “Visualization of Seifert Surfaces.” IEEE Trans. Visualization & Computer Graphics, Vol. 1, No. X, Aug. 2006. – https://www.maths.ed.ac.uk/~v1ranick/papers/vanwijk.pdf