A Tale of Three p4g Origami Models and Local Frieze Symmetry

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Abstract

The origami "switchboard" is a rectangular array of square twists, each viewed as a "switch" that can be configured in 18 different ways. The article proposes the use of the switchboard to create origami models of patterns with rectangular geometry and to explore their properties. The approach is demonstrated for three variants of $p4g$ wallpaper patterns.

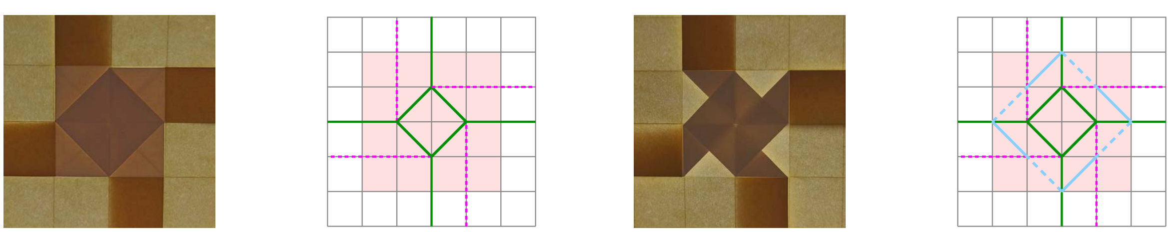
Introduction

This paper has two themes, developed in parallel. The first is the design and construction of adaptable flat-folded origami models, with the purpose of demonstrating specific geometric features of particular patterns. The origami artworks in this article are all hand-folded by the author, from an uncreased square sheet.

The second theme is the study of wallpaper patterns of the $p4g$ class. We consider such a pattern as a union of parallel strips, each bounded by a reflection axis and a glide axis of the wallpaper pattern, and identify three distinct subtypes of $p4g$ patterns. This theme is developed in the second and third sections of the article, and the two are combined in the final section. For background on the wallpaper classification, on relevant origami methods, and on the mathematics of tessellation origami, we refer to [1], [2] and [3].

The Switchboard: a Canvas for Adaptable Origami Wallpaper Models

The *square twist* (Figure 1) is a standard, elementary and beautiful move in geometric origami. The starting point for folding it is a precreased square grid. In Figure 1(a), the edges of the dark foreground square lie over diagonals of square cells in the original grid, and the area of this *twisted square* is double that of a grid square. The twist compresses sixteen squares of the original grid, corresponding to the pink squares in the crease pattern (b), to the highlighted 2×2 square at the centre of the folded form in (a). The 20 additional grid squares that surround this region in (b) are reduced to 12 in the folded form. This surrounding area is shown in order to highlight to four pleats of width 1 that emanate from the twist. Solid and dashed lines in the crease diagrams respectively denote mountain and valley folds. The square twist has a clockwise and counter-clockwise version. They are mirror images of each other in a horizontal or vertical axis.

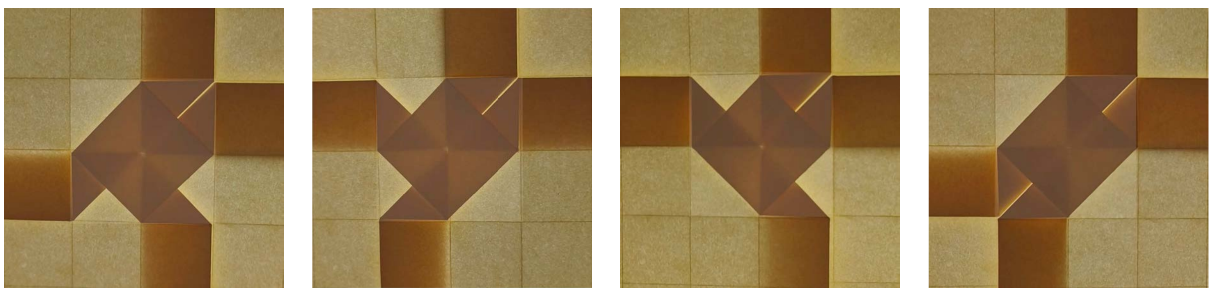


(a) *Square twist* **(b)** *Twist crease pattern* **(c)** *Square untwist* **(d)** *Untwist crease pattern*

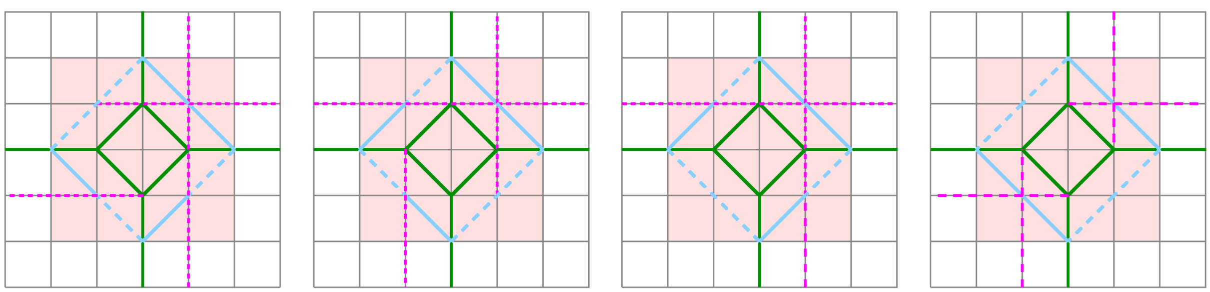
Figure 1: *The square twist and square untwist*

The term *twist* refers to the fact that the dark foreground square in Figure 1(a) is twisted with respect to its position in the unfolded sheet; each of its edges is orthogonal to the orientation of the same line when the paper is unfolded. The square twist can be adapted as follows. We can undo the twisting by introducing the valley and mountain folds respectively indicated in broken and solid light blue lines in Figure 1(d). The effect on the physical structure of Figure 1(a) is less complicated than the crease pattern might suggest: it is to introduce four diagonal folds, and to "untwist" the foreground square of Figure 1(a) while preserving its overall position. We refer to the configuration of Figure 1(c) (or its mirror image) as the *square untwist*.

The square untwist is a remarkably versatile element in an origami model. It is an initial adaptation of a square twist that admits further simple adjustments, affecting the geometry of its local environment. Each of the four triangular pleats of Figure 1(c) may be *reoriented* by temporarily lifting the foreground square and rotating the pleat about its base under it, like turning a page. The photographs in Figure 2 show the possible outcomes of (combinations of) such moves, up to a rotation. The number of distinct configurations attained by rotating the pictured one is noted in each caption. The original twist and untwist, and their mirror images, bring the total number of configurations arising from a single square twist to 18.



(a) *4 rotations* **(b)** *4 rotations* **(c)** *4 rotations* **(d)** *2 rotations*

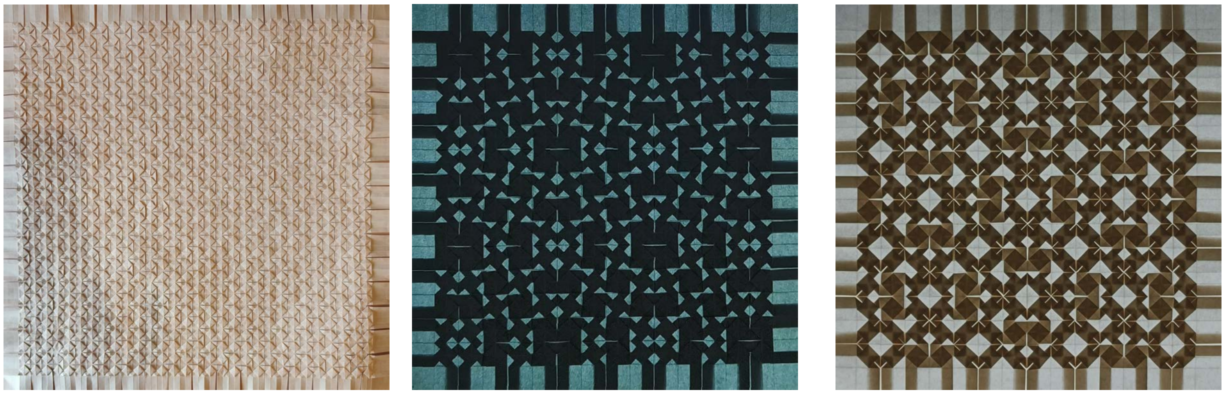


(e) *Crease pattern for (a)* **(f)** *Crease pattern for (b)* **(g)** *Crease pattern for (c)* **(h)** *Crease pattern for (d)*

Figure 2: *Variations on the square (un)twist*

We define a *switch* to be a square twist (or untwist) that is configurable in the 18 ways described above. In the photographs of Figures 1 and 2, the switch is the highlighted central 2×2 region, corresponding to the area of the crease diagram with pink background. A *switchboard* is a rectangular array of non-overlapping switches, each occupying a 4×4 region in the underlying precreased grid. Figure 3(a) shows a 30×30 switchboard with 900 square twists, which is the basis for the three *p*4*g* models in the final section. Due to the relative ease of folding square twists on a precreased grid, it is convenient to craft new switchboards in this initial form. The *cmm* model in Figure 3(b) is folded on a 14×14 switchboard. Of the 18 possible switch configurations it uses only the square twist and untwist, in both mirror image orientations. The *p*4*g* model in Figure 3(c) is also folded from a 14×14 switchboard, and its switches have 12 of the 18 configurations.

To fold a $m \times n$ switchboard requires a $4m \times 4n$ precreased square grid. Figure 3 is intended to demonstrate some of the versatility and scope of the switchboard design as a basis for origami tessellations



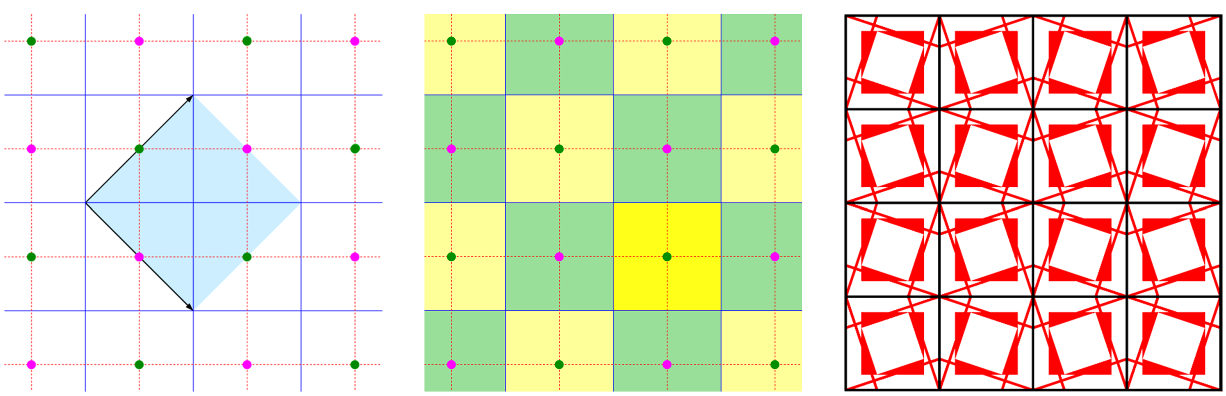
(a) 30 × 30 *Switchboard* **(b)** *cmm Switchboard (*14 × 14*)* **(c)** *p4g Switchboard (*14 × 14*)*

Figure 3: *Origami Models based on the Switchboard*

that are adaptable in the sense that their individual switches admit the multiple configurations shown (up to rotation) in Figures 1 and 2. While each switch in a switchboard can be configured in 18 distinct ways, neighbouring switches cannot be configured entirely independently of each other. Any pair of adjacent switches share a pleat that connects them, and the orientations of this pleat in the two switches must align. For example, the switches of Figure 2(c) and (d) could not occur side by side in a switchboard as they appear in Figure 2, since the positions of the right pleat in (c) and the left pleat in (d) are incompatible.

The p4g Wallpaper Class

A *p*4*g* wallpaper pattern has reflection axes in two perpendicular directions, with glide reflection axes in the same two directions, located midway between pairs of adjacent reflection axes. Centres of four-fold rotation are located (only) at the intersection points of glide axes. Figure 4(a) shows the arrangement of orthogonal reflection axes (solid blue) and glide axes (dashed red). The lattice of translational symmetries is generated by the pair of orthogonal translations shown as black arrows. The light blue square is a translation unit cell. The action of group of translations on the set of four-fold rotation centres has two orbits, whose elements are respectively coloured pink and green. The reflection in any blue axis interchanges these two orbits.



(a) *p*4*g axes and translations* **(b)** *p*4*g checkerboard* **(c)** *a p*4*g pattern*

Figure 4: *Description and example for the p*4*g wallpaper pattern class*

Let S be the square region bounded by adjacent reflection axes in the horizontal and vertical directions, shown in dark yellow in Figure 4(b). Independently of the wider pattern, we may regard S as a geometric object and consider its symmetry group, which is a subgroup of the dihedral group D_8 of order 8. Since the 90° rotation about the centre of S is a symmetry of the entire wallpaper pattern, it is a symmetry of S, and the symmetry group of S contains the cyclic group of order 4 generated by this rotation. Each green square in Figure 4(b) is a translated copy of the mirror image of S in a horizontal or equivalently a vertical axis. The $p4g$ pattern is a checkerboard of copies of S and its reflection S'. It follows that the symmetry group of S consists only of rotations. The only alternative is that S has full D_8 symmetry, which would mean that the yellow and green squares of Figure 4(b) would be identical, and the pattern would belong to the class $p4m$.

These observations characterize $p4g$ patterns: a $p4g$ wallpaper pattern is a checkerboard tiling of the plane, with a square tile S whose symmetry group is cyclic of order 4, and its mirror image S'. Any square tile with four-fold rotational symmetry (and no reflection) generates a $p4g$ pattern through this checkerboard construction. Figure 4(c) shows an example of a $p4g$ pattern that is intended to highlight this viewpoint. We will refer to the square region bounded by adjacent reflection axes in a $p4g$ pattern as a *slab*. A slab in a $p4g$ pattern is divided into four squares by the horizontal and vertical glide axes that pass through its centre. Each of these squares is bounded by a pair of reflection axes and a pair of glide axes, and all four are images of each other under rotations about the centre of the slab. We refer to a square of this type in a $p4g$ pattern as a *slate*. The term is intended to evoke both the idea of a tile and of something that can be wiped clean and rewritten. This last feature is relevant to the design of origami $p4g$ patterns with designated properties, which will be discussed in the final section. In the representation in Figure 4(b) of a general $p4g$ pattern, the green and yellow squares represent the slabs. Each is visibly divided into four slates, by the glide axes that intersect at the centre of the slab, which is a four-fold rotation centre. The pattern is a union of slates that intersect only on their boundaries, and all slates are images of each other under wallpaper group elements. A slate is a fundamental domain for the wallpaper group action.

Intrinsic Friezes in p4g Patterns

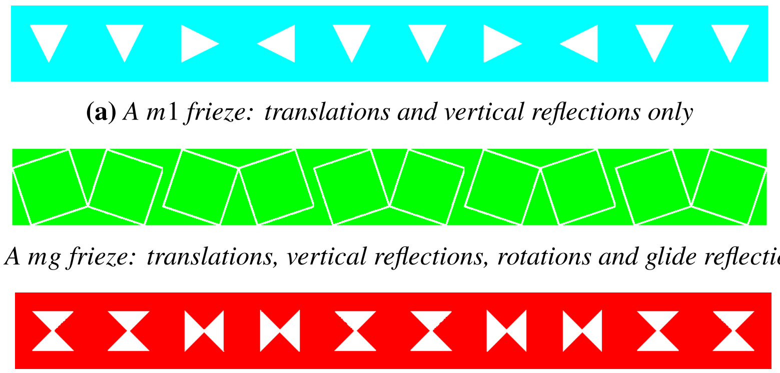
Within a given $p4g$ pattern P, we consider the strip F bounded by a reflection axis and a glide axis that are parallel and adjacent. This region is a frieze, since it has translational symmetries that are restrictions to F of translational symmetries of the wallpaper pattern in a direction parallel to F. On the other hand, since the entire $p4g$ pattern is a union of parallel copies of F and its mirror image, every translational symmetry of F extends to a translational symmetry of P. Thus the group of translational symmetries of the frieze F is an infinite cyclic subgroup of the group of translational symmetries of P.

We refer to F as an *intrinsic frieze* in P. This terminology was introduced in [4], in the context of the question of which frieze pattern classes can be represented by strips within a pattern of a specified wallpaper class. This question is quite loose in its general form. It can be sharpened by restricting attention to strips that are determined by intrinsic features of the geometry of the wallpaper pattern, such as neighbouring parallel reflection and glide axes. We now consider the possible symmetry structures of intrinsic friezes in $p4g$ patterns.

It can be assumed that F runs in the horizontal direction, since intrinsic friezes in the vertical orientation are images of the horizontal ones under rotational symmetries of P. The symmetry group of F includes reflections in the vertical reflection axes of P. These reflections are symmetries of P that preserve the subset F, hence they restrict to symmetries of F.

Since the centre line of F is not a reflection axis or glide axis of P, and does not contain any two-fold rotation centres of P, any frieze symmetry of F that extends to a wallpaper symmetry of P is either a horizontal translation or a reflection in a vertical axis. On the other hand, F may be regarded as a geometric object in its own right, independently of the ambient wallpaper pattern. This object may have additional symmetries that do not extend to symmetries of P and are not detected by the wallpaper group. For example, the centre line of F might be a reflection axis of F but not of P, or some two-fold rotation could be a symmetry of F but not of P. Any such features are *local* to F, in the context of the situation of F as a subset of the wallpaper pattern.

Since the symmetry group of F includes vertical reflections, its frieze pattern class is one of the three that includes vertical reflections: $m1, mg$ and mm. These are described by means of examples in Figure 5.



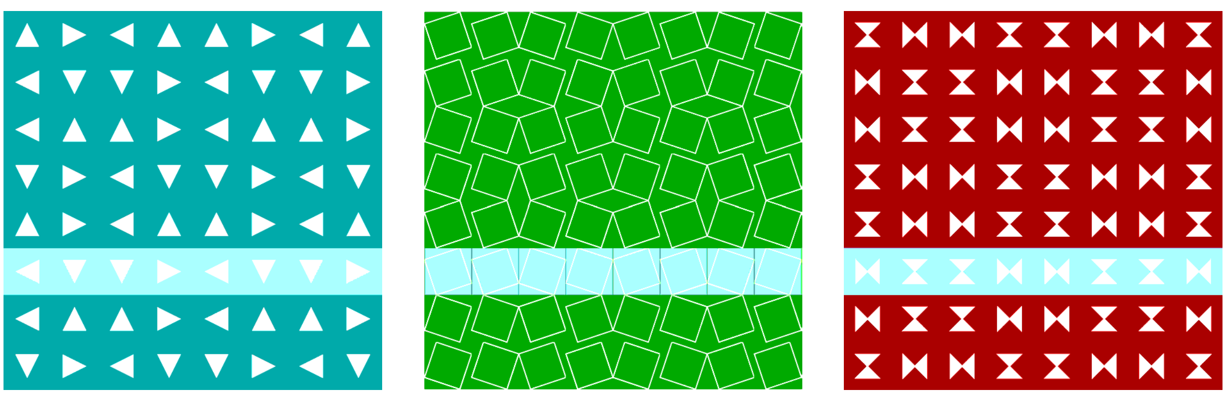
(a) *A* $m1$ *frieze: translations and vertical reflections only*

(b) *A* mg *frieze: translations, vertical reflections, rotations and glide reflections*

(c) *A* mm *frieze: translations, vertical reflections, horizontal reflection and rotations*

Figure 5: *The frieze pattern classes with vertical reflections*

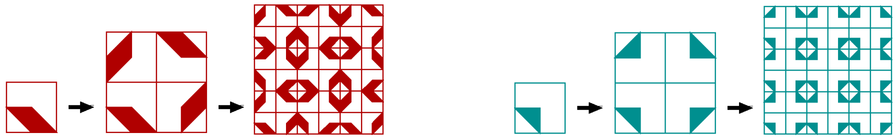
That each of the three can arise as the frieze pattern class of the intrinsic friezes of a $p4g$ pattern is confirmed by the examples in Figure 6 below. We say that a $p4g$ wallpaper pattern has *subtype* $m1, mg$ or mm according to the frieze pattern class of its intrinsic friezes, and denote the three subtypes by $p4g[m1]$, $p4g[mg]$ and $p4g[mm]$.



(a) *A* $p4g[m1]$ *pattern* **(b)** *A* $p4g[mg]$ *pattern* **(c)** *A* $p4g[mm]$ *pattern*

Figure 6: $p4g$ *patterns of the three subtypes*

Every $p4g$ pattern belongs to one of the three subtypes, determined by identifying a reflection axis and an adjacent glide axis and observing the frieze pattern class of the strip between them. To discuss the construction of models of all three types, particularly from the origami switchboard, we return to slates and slabs in $p4g$ patterns. Suppose that L is a given square tile. We ask whether L can occur as a slate in a $p4g$ pattern, with reflection axes along its left and lower edges, and glide axes on its right and upper edges. If so, the pattern is extended from L as follows. We construct the square region S comprising L and its images under rotations through 90°, 180° and 270° about its upper right vertex. We note that S has four-fold rotational symmetry about its centre. We form a checkerboard pattern of S and its (horizontal or vertical) reflection S'. Figure 7 shows two examples of the progression from L to S to the pattern.



(a) *p4g pattern from a slate* **(b)** *p4m pattern from a slate*

Figure 7: *Two examples of pattern construction from a slate*

The construction produces a $p4g$ pattern if the symmetry group of S has order 4 and consists only of rotations, and it produces a $p4m$ pattern if S has the full dihedral symmetry group of the square. The latter case occurs if and only if reflection in the diagonal through the centre of S is a symmetry of the slate L. The slates of Figure 7 demonstrate both scenarios.

Suppose now that L is a slate for which the above construction leads to a $p4g$ pattern P. The symmetry group of L has order 1, 2 or 4, since it contains at most one reflection in a diagonal. This symmetry group determines the $p4g$-subtype of P as follows.

An intrinsic frieze in P is a strip of slates. We write ⌟ for the image of L under a 90° counterclockwise rotation, and we write ⌞ and ⅃ respectively for the images of ⌟ and L under a reflection in a vertical axis. The arrangement of slates along a horizontal intrinsic frieze repeats the sequence

L ⌟ ⌞ ⅃ L ⌟ ⌞ ⅃

The minimum translation length of an intrinsic frieze is the the width of two slabs, or four slates.

1. The frieze has type mm if and only if it is symmetric under reflection in a horizontal axis, which means that L, ⌟, ⌞ and ⅃ are all symmetric under such a reflection. This occurs if and only if the symmetry group of the slate L is the Klein 4-group generated by the horizontal and vertical reflections.
2. The frieze has type mg if and only if the composition of horizontal reflection with translation by two slate lengths is a symmetry of the frieze. This means that the image of L under reflection in a horizontal axis is ⌞, which occurs exactly if the slate L is preserved by rotation through 90°. The $p4g$ pattern P has subtype $[mg]$ if and only if the symmetry group of the slate L is cyclic of order 4.
3. The $p4g$ pattern P has type $m1$ if the symmetry group of the slate L is trivial or has order 2.

Figure 8 shows slates for the three patterns in Figure 5, each with the required symmetry for the pattern subtype, and a slate for the origami model of Figure 3(c), which has type $m1$.

(a) *A* $p4g[m1]$ *slate* **(b)** *A* $p4g[mm]$ *slate* **(c)** *A* $p4g[mg]$ *slate* **(d)** *A Figure 3(c) slate*

Figure 8: *Slates for the* $p4g$ *patterns of Figures 5 and 3*

Origami Models from the 30 × 30 Switchboard

This final section presents origami models of $p4g$ patterns of the three subtypes, shown in Figure 10, all folded on a 30×30 switchboard as in Figure 3(a). For comparability and transition via switch adjustments, the locations of the reflection and glide axes, and slates and slabs, are the same in all three models.

For each model, the fundamental component is a slate with the symmetry group satisfying the requirements for the relevant subtype. The size of the slate must be sufficient to allow its symmetries to be managed and altered by switch reconfigurations in way that is visible to a viewer, but small enough that a switchboard of feasible size can contain enough slabs to make the $p4g$ pattern and its periodicity recognizable.

There is an additional compatibility constraint involving edges between adjacent slates. If a slate is a $k \times k$ array of switches, then it has k switches along each of its four edges. From each of these k switches, there is a pleat crossing the edge, that occupies half of the width of the switch. Along the edge between two adjacent slates, these pleat positions must match in a physically constructible model. Throughout a $p4g$ model, slates are adjacent along edges to their rotated and reflected copies. One way to satisfy the boundary crossing constraint is to ensure that the arrangement of pleats crossing the boundary is preserved under all rotations and reflections of the square.

A slate for a $p4g[mm]$ model has a Klein 4-group of symmetries, generated by reflections in its horizontal and vertical midlines. This requires that the slate is constructed on a $k \times k$ switchboard with k even, since horizontal and vertical reflection axes cannot bisect switches. Figures 1 and 2 show that no configuration of a single switch has a horizontal or vertical reflection as a symmetry. The first possibility is $k = 2$, but this is limited in scope. The next option for the size of a slate is a 4×4 array of switches. This is amenable to the symmetry requirements of all three pattern types. Figure 9 shows origami slates for the models, with the required Klein 4-group and cyclic group of rotations as the symmetry groups for the subtypes mm and mg. The $p4g[m1]$ slate has a symmetry group of order 2, generated by a reflection. All three have the same arrangement of boundary pleat crossings, invariant under all rotations and reflections of the square.

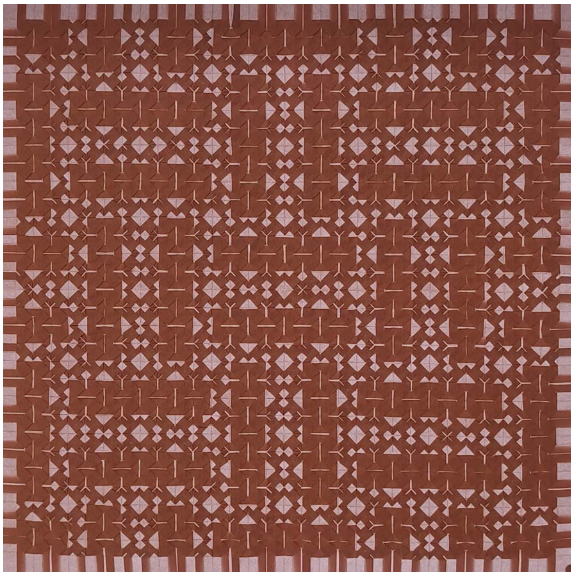
(a) *A* 4×4 *switchboard* **(b)** *A* $p4g[mm]$ *slate* **(c)** *A* $p4g[mg]$ *slate* **(d)** *A* $p4g[m1]$ *slate*

Figure 9: *Slates from a* 4×4 *switchboard, for* $p4g[mm]$, $p4g[mg]$ *and* $p4g[m1]$ *subtypes*

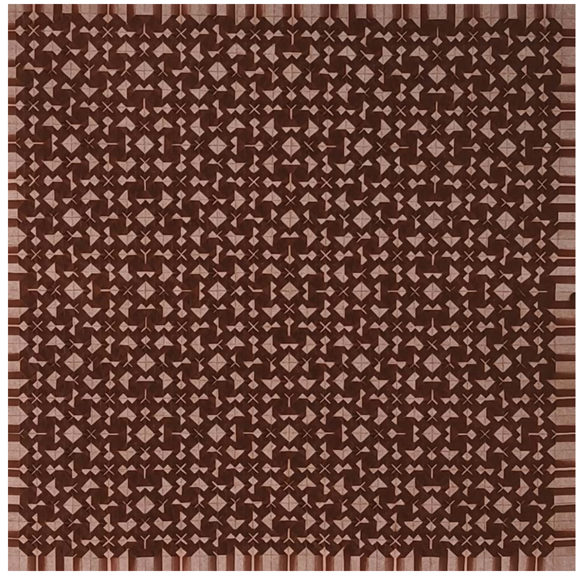
In the construction of physical models from these slates, some further practicalities arise. A precreased square grid on a square sheet is essentially constrained to size $n \times n$ where n is a power of 2, or perhaps slightly less than a power of 2, if some space is left at the edges. A slate requires a 4×4 array of switches, or a 16×16 section of the precreased grid. A single slab occupies a 32×32 precreased grid, so a 64×64 grid is not sufficient for a visual display of a $p4g$ pattern based on 16×16 slates. The models of Figure 10 are adapted from the 30×30 switchboard of Figure 3, folded from a 120×120 precreased grid. To fold a switchboard of this size by hand is a substantial undertaking, involving 900 square twists, on a 70cm square sheet in this case. Fortunately there is a reward in the scope and adaptability of the resulting canvas. An intrinsic frieze from each model is shown in Figure 10 (d), displaying the mm, mg and $m1$ frieze symmetry.

Concluding Remarks

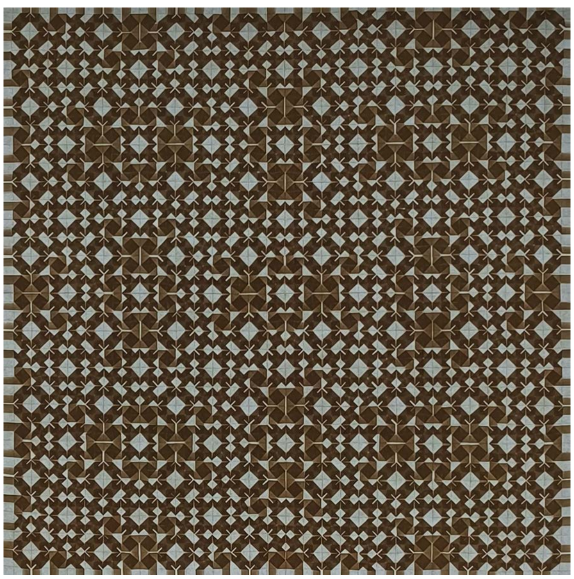
The square twist switchboard can be used to model wallpaper patterns from any of the twelve classes whose symmetry groups do not include 3-fold rotations, and to explore variations within the pattern class. The design of an analogous switchboard for wallpaper patterns from the five classes with 3-fold rotations is more challenging, and is a subject for future attention.



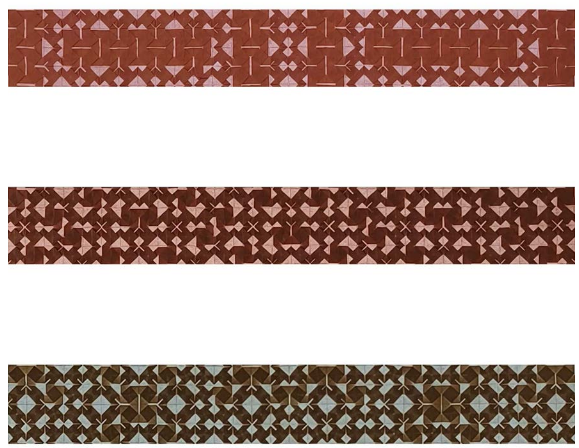
(a) *Origami model of a* $p4g[mm]$ *pattern*



(b) *Origami model of* $p4g[mg]$ *pattern*



(c) *origami model of* $p4g[m1]$ *pattern*



(d) *Intrinsic friezes in (a), (b) and (c)*

Figure 10: *Models of* $p4g[mm]$ *and* $p4g[mg]$ *patterns from a* 30×30 *switchboard*

References

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