Tiling a Hyperbolic Pair of Pants

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Abstract

A hyperbolic regular octagon with 45° interior angles can be edge-identified to construct a double torus, analogous to how a square can be edge-identified to form a torus. A pair of pants arises as an intermediate step in this process, much like how a cylinder appears as an intermediate step when constructing a torus. In this paper we show how to build a regular octagon with 45° interiors angles using the hyperbolic $(6, 6, 8)$ tiling. The final model is made of EVA foam pieces which allow easily constructing a pair of pants from the octagon.

Introduction

It is easy to visualise how a square can be edge-identified to form a doughnut shaped torus. Starting with a square, we first glue the top and bottom edges together to create a cylinder and then stretch one edge of the cylinder to meet the other, completing the torus, as shown in Figure 2. Constructing a double torus through a similar process is a bit more complicated. We begin with a regular octagon, and instead of forming a cylinder, we first create a *pair of pants* by gluing two pairs of edges together. To form the double torus, we then pinch the waist of the pants to create two loops and, similarly to when creating a torus, stretch and join the two 'pant legs' to the corresponding loops in the middle. This process is illustrated in Figure 1.

The last step is where we encounter a problem. The interior angles of a square are 90°, so when four of them meet to form a torus, they neatly add up to 360°. The interior angles of a regular octagon, however, are 135°, and when eight of them meet, they sum to 1080°, which does not result in a smooth surface. To construct a smooth double torus, we need an octagon with 45° interior angles. Such an octagon cannot exist in the Euclidean plane, but it does in hyperbolic space.

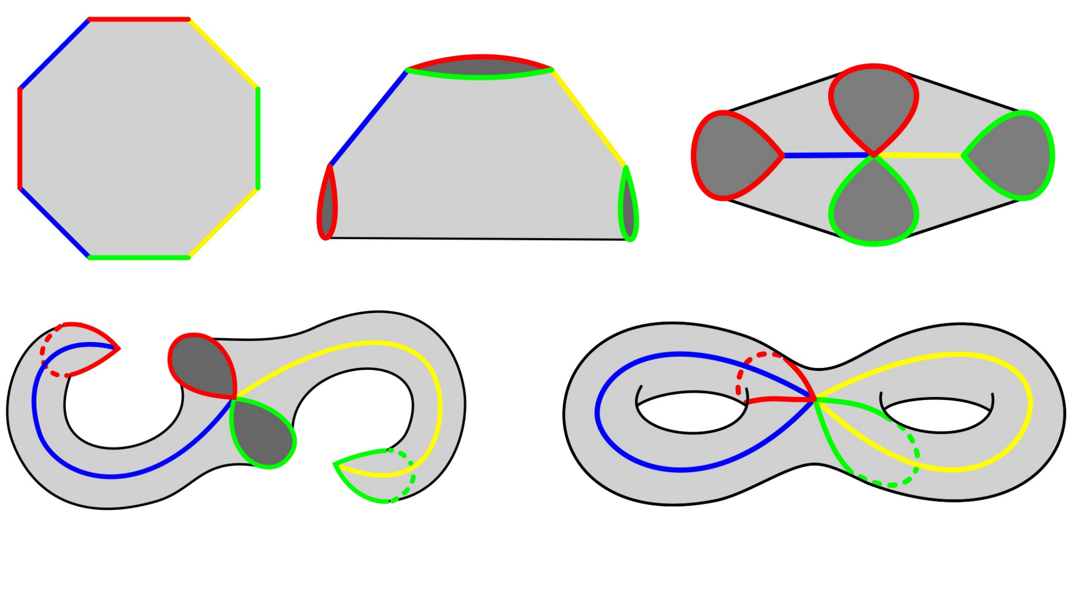

Figure 1: *The double torus can be constructed from an octagon by identifying the edges pairwise. A pair of pants appears as an intermediate step (top row middle).*

So how can we create a physical model of a hyperbolic octagon with 45° interior angles? One approach, introduced in [5], is to 'draw' the octagon on a crocheted hyperbolic surface, with the angles adjusted by trial and error. In this paper, we will use the Gauss-Bonnet theorem to construct an octagon with 45° interior angles, based on the hyperbolic $(6, 6, 8)$ tessellation. The physical model is created using Curvagons, introduced in [3], which allow for a straightforward construction of a pair of pants from the octagon.

From a Square to a Doughnut

Before we move to the double torus and the pair of pants, we begin with the torus and see how it can be constructed by 'gluing' the opposite edges of a square together. We define the flat torus as the square $R = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ with the following equivalence relations on its edges:

- Identify the left and right edges (red in Figure 2)

$$(0, y) \sim (1, y) \quad \text{for all } y \in [0, 1].$$

- Identify the top and bottom edges (blue in Figure 2)

$$(x, 0) \sim (x, 1) \quad \text{for all } x \in [0, 1].$$

This means that the square is 'wrapped around' so that opposite edges are connected. The flat torus is like the classic Snake game; when the snake exits the screen on the right (or top), it reappears at the corresponding point on the left (or bottom), and vice versa. It is important to note that the flat torus is an abstract mathematical object that cannot be smoothly embedded in $\mathbb{R}^3$ without distortion.

Next, we can imagine transforming a square piece of rubber into a doughnut, which we will call the ring torus, by joining the opposite edges of the square. First, take the square and roll it so that the top edge meets the bottom edge, forming a cylinder. Next, we need to imagine that the square is made of very flexible rubber. Bend the cylinder around so that the right edge meets the left edge, as illustrated in Figure 2. Note that the interior angles of the square are 90°, so when four of them meet, they sum to 360°. Since the square has ordinary Euclidean (flat) geometry, we say that the torus can be equipped with a Euclidean structure.

The flat torus and the ring torus are both closed, orientable surfaces that can be described topologically as the product of two circles, $S^1 \times S^1$. This means they each have two non-separating loops, generating the fundamental group of the surface, shown in red and blue in Figure 2. We can also explore the topological structure of the torus by examining its Euler characteristic. The Euler characteristic is a topological invariant that remains unchanged regardless of how the space is bent or embedded and thus gives a way to determine topologically equivalent surfaces. For a closed, orientable surface, the Euler characteristic is given by

$$\chi = 2 - 2g$$

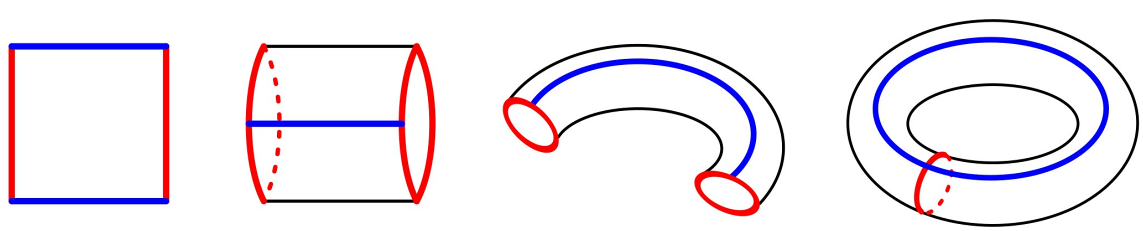

Figure 2: *A square can be edge-identified to form a torus.*

where g is the genus (the number of doughnut-like holes) of the surface. Since the genus of the ring torus is one, we see that its Euler characteristic is zero. For polyhedra and graphs, the Euler characteristic can be expressed as

$$\chi = V - E + F,$$

where V is the number of vertices, E the number of edges, and F the number of faces. It is easy to see that the number of faces of the flat torus is one. Since we identified the left and right edge as one, and the top and bottom edge as one the number of edges is two. Looking at the definition of the flat torus we notice that all the vertices are identified as one $(0,0) \sim (1,0) \sim (1,1) \sim (0,1)$. We can conclude that, indeed, $\chi = 1-2+1 = 0$ for the flat torus. Note that while the flat torus looks like an ordinary square, it is topologically very different. A square has one face, four edges, and four vertices, resulting in $\chi = 1$.

An old mathematics joke goes that a topologist is someone who cannot tell the difference between a coffee cup and a doughnut. The same could be said about the flat torus and the ring torus. However, the two are clearly different geometrically. The flat torus has zero *Gaussian curvature* everywhere, while the ring torus has varying Gaussian curvature. The region on the outer edge of the torus has positive Gaussian curvature, as the surface bends in the same direction in all directions, much like a sphere. At the innermost part of the torus, near the hole, the Gaussian curvature is negative, since the surface bends in opposite directions, resembling the shape of a saddle, see Figure 3.

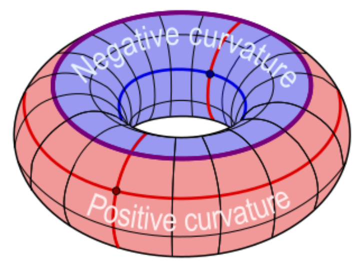

Figure 3: *The torus has varying curvature.*

There appears to be some disconnect between geometry, which focuses on local properties such as curvatures, angles and areas, and topology, which studies properties that remain unchanged regardless of exact geometric measurements. Some of this gap can be bridged through the Gauss-Bonnet theorem. This fundamental result connects the geometrical concept of curvature with the topological concept of the Euler characteristic. The Gauss-Bonnet theorem states that for a compact surface M with a smooth boundary ∂M, and geodesic curvature of the boundary k, the following holds:

$$\int_M K\, dA + \int_{\partial M} k\, ds = 360^\circ \cdot \chi, \tag{1}$$

where K is the Gaussian curvature of the surface, dA is the surface area element, and ds is the line element along the boundary. For compact surfaces without boundary the second integral vanishes and the theorem simplifies to state that the *total Gaussian curvature* of a closed surface is $360^\circ \cdot \chi$. This shows that although the flat torus and the ring torus have different Gaussian curvatures, their total Gaussian curvature is zero. In general, the Gauss–Bonnet theorem implies that no matter how a surface is bent, as long as the surface stays topologically the same (no holes added or removed), the total curvature stays constant, because the Euler characteristic (being a topological invariant) does not change.

From an Octagon to a Double Torus

The double torus has two doughnut-like holes, or handles, and it can be constructed from an octagon by identifying its edges in pairs. There are several ways to glue the edges together to form a double torus, but we will focus on the gluing pattern that results in a pair of pants as an intermediate step. Similar to when constructing a torus from a square, all the corners of the octagon meet at one point. Since the interior angles of a regular flat octagon are 135°, the sum of the angles at the vertex is 1080° instead of 360°. If we attempt to assign the double torus the same geometric structure as the flat octagon, we end up with a double torus having Euclidean geometry everywhere except at one point, where a singularity occurs. We want to create a smooth geometric structure including the vertex. This means that we need an octagon whose interior angles are 45° instead of 135°. Clearly such an octagon cannot exist in the Euclidean plane.

Here we need to move from the familiar Euclidean plane to the hyperbolic plane. The hyperbolic plane has constant negative Gaussian curvature. The interior angles of an octagon drawn in the hyperbolic plane add up to less than 1080°. The difference depends on the size of the octagon. For very small octagons, the angles sum to nearly 1080°, but as the octagon grows larger, the angles become smaller. The largest octagon we can draw has vertices stretching towards infinity, with interior angles approaching zero. This happens because on the hyperbolic plane parallel lines (lines that do not intersect) can diverge from each other rapidly. As a result, attempting to draw an even larger octagon would mean the sides never meet to form a vertex. Between the two extremes, there exists an octagon whose interior angles are exactly 45°, as shown in the Poincaré disk model in Figure 4.

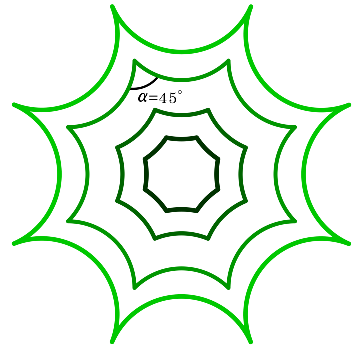

Figure 4: *Hyperbolic octagons.*

Next, we identify the edges of the octagon in pairs. In Figure 1, we first glue the blue and yellow edges pairwise, resulting in a pair of pants. A pair of pants is a two-dimensional surface with three boundary components — the 'waist' and two 'leg openings'. Topologically, it is equivalent to a sphere with three open disks removed. Any orientable closed surface of at least genus 2 can be decomposed into a finite collection of pairs of pants using simple closed curves. This pants decomposition simplifies the study of complex surfaces by breaking them into well-understood components, providing a powerful tool for understanding and classifying surfaces. The pair of pants decomposition has many applications in mathematics and physics, from topology and algebraic geometry to quantum field theory and string theory. See [1] for a brief description of the use of pairs of pants in mathematics, physics, and ceramics.

To create a double torus from a pair of pants, we pinch the left and right corners of the 'waist' together to form two loops, as shown in the top right of Figure 1 (viewed from top of the pants). Next, we need to imagine that the octagon is made from a very stretchy material. To close the double torus, we bend the left and right pant legs to meet the loops in the middle. The resulting double torus can be equipped with the hyperbolic structure of the hyperbolic octagon.

Since the double torus has two doughnut-like holes, it is a genus 2 surface, meaning its Euler characteristic is -2. We can also use the second definition of the Euler characteristic. Having identified the sides of the octagon in pairs and with all the vertices meeting at one point (i.e., they are identified as one), we have one face, four edges, and one vertex, which gives us $\chi = 1 - 4 + 1 = -2$. Using the Gauss-Bonnet theorem, we can then conclude that the total curvature of the double torus is $-720°$.

We can use similar construction to create surfaces with $g \geq 3$ handles by first gluing together $2g$ sides of a polygon to form g legs, then pinching the middle to create g loops, and finally gluing the legs to the middle. This process shows that a surface of genus $g \geq 2$ can be constructed from a $4g$-gon with interior angles of $360°/4g$. These surfaces can then be equipped with the hyperbolic structure of the hyperbolic $4g$-gon. We conclude that closed surfaces can be divided into three groups: the sphere, which has spherical structure: the torus, which can be constructed from a flat square and thus admits Euclidean structure: and infinitely many genus $g \geq 2$ surfaces, which can be build from hyperbolic $4g$-gons and therefore equipped with hyperbolic structure.

Creating an Octagon with 45° Angles

As discussed in the previous section, we can draw a regular octagon with interior angles $0° < \gamma < 135°$ on a hyperbolic surface. But how can we construct an octagon with exactly 45° interior angles? One method, as shown in [5], is to crochet a large piece of hyperbolic surface and then stitch the octagon onto it. The correct shape was found through trial and error by testing different octagon sizes. The crochet approach leaves excess hyperbolic surface around the octagon, as the surrounding area cannot simply be trimmed away. We will explore how to construct a hyperbolic octagon and a pair of pants using a tessellation model of the hyperbolic plane. The model we introduce is fast and easy to construct and results in an octagon with precisely 45° interior angles.

The Euclidean plane, sphere, and hyperbolic plane have constant Gaussian curvatures. Since their curvature is uniform at every point, they can be easily approximated using simple tessellations. Consider an equilateral triangle, where the interior angles are 60°. If you place six of these triangles at a vertex, their angles add up to $6 \cdot 60° = 360°$, creating a flat surface — a piece of the Euclidean plane. If you place only five triangles at a vertex, the surface will curve in on itself. As the tessellation continues, it will close, forming a crude approximation of a sphere, known as the icosahedron. With five triangles, the angles at a vertex add up to 300°, resulting in a $360° - 300° = 60°$ angle defect. Positive angle defect leads to a an approximation of a surface with positive Gaussian curvature. On the other hand, if you place seven triangles at a vertex, the surface will form a saddle-like shape. In this case, the angles add up to 420°, creating a $-60°$ angle defect which we will call angle excess. Angle excess leads to an approximation of a surfaces with negative Gaussian curvature — a piece of the hyperbolic plane.

The triangle-based model of a hyperbolic surface is quite wrinkly due to the 60° angle excess at each vertex. Smoother approximations of the hyperbolic plane can be achieved by using tessellations with smaller angle excess. One such model is the hyperbolic football tessellation where two hexagons and a heptagon meet at every vertex and the angle excess is only 8.57° [2, 4]. For the hyperbolic octagon with 45° interior angles we would like to create a model that is as symmetric and smooth as possible. For this, we use the $(6, 6, 8)$ tessellation, where one octagon and two hexagons meet at each vertex, producing a 15° angle excess.

Since the curvature of a tiling model is concentrated at the vertices, knowing the total curvature of the hyperbolic octagon would allow us to determine how many vertices must lie within the model. The total Gaussian curvature of a polygon is closely related to the sum of its interior angles through the Gauss-Bonnet theorem. For an n-polygon M we can interpret the second integral in (1) as integrals along the edges plus the turning of the edges at the vertices. Since the edges of a polygon are geodesics (locally length-minimising curves), their geodesic curvature is zero, and it is enough to consider the total turning

$$\Gamma = \sum_{i=1}^{n} (180° - \alpha_i),$$

where α_i are the interior angles and $180° - \alpha_i$ are the turning angles. Using the fact that the Euler characteristic of any polygon is 1 we can conclude that the total Gaussian curvature of an n-polygon M is

$$\int_M K \, dA = 360° - \Gamma.$$

For a regular hyperbolic octagon with interior angles of 45°, the turning angles are 135°. Using the above we see that the total curvature of such an octagon is $360° - 8 \cdot 135° = -720°$. To construct this octagon using the $(6, 6, 8)$ tessellation, we observe that each vertex of the tessellation contributes $-15°$ to the curvature. Therefore, the octagon must contain $720/15 = 48$ vertices. To keep the model as

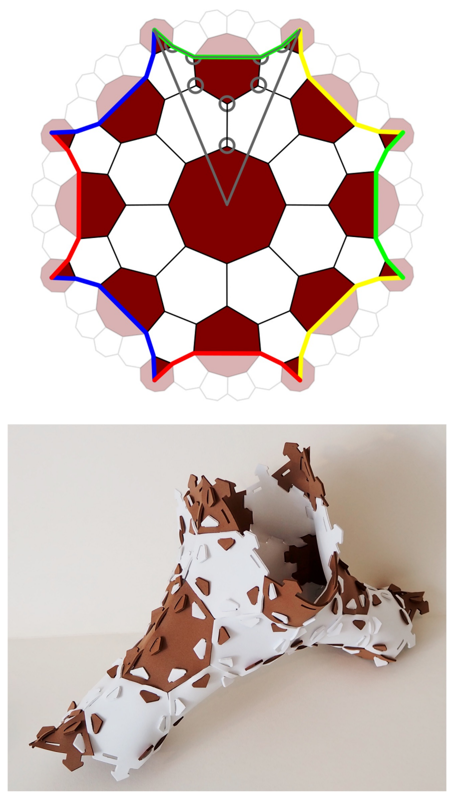

Figure 5: *Tesselated hyperbolic octagon and pair of pants.*

simple as possible we aim to create a model consisting of eight identical hyperbolic triangles. This means the total curvature of a triangle is 90° and each triangle should contain 6 vertices.

The hyperbolic regular octagon with 45° interior angles on the $(6, 6, 8)$ tessellation is illustrated at the top of Figure 5. One of the eight triangles is drawn at the top, with the vertices of the tessellation circled in grey. Four vertices lie inside the triangle, while two are halved along the edges of the hyperbolic octagon. Each corner of the hyperbolic octagon consist of one-eighth of a flat octagon tile. When constructing a pair of pants, the halved flat octagon tiles at the edges of the hyperbolic octagon are joined together, as can be seen at the bottom of Figure 5, resulting in vertices with a 15° angle excess. A pair of pants with the paired edges coloured is shown in Figure 6. The corners of the 'waist' are pinched together to create the loops in the middle to which the 'leg openings' should be glued. You can make your own model by cutting out 6 octagons and 16 hexagons from paper, then halving 4 of the octagons and cutting one of them into 8 equal pieces. Assemble the pieces with tape, following the layout shown in Figure 5.

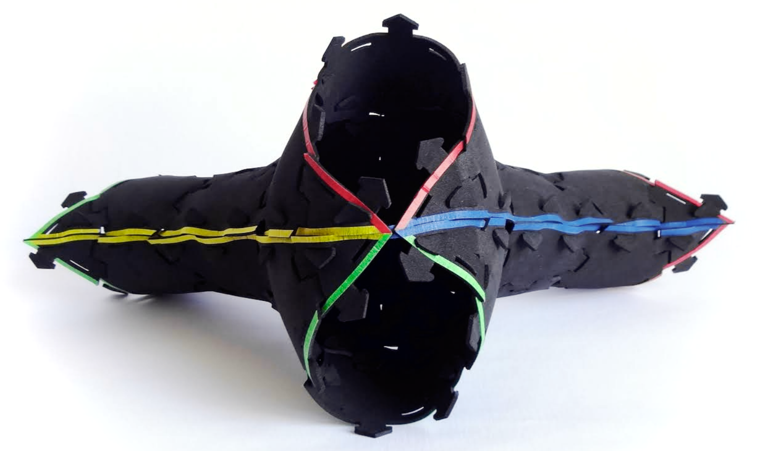

Figure 6: *A pair of pants with the 'waist' pinched together to create two loops in the middle.*

Summary and Conclusions

A hyperbolic regular octagon with 45° interior angles can be edge-identified to form a double torus, with a pair of pants appearing as an interesting intermediate step in the process. In this paper, we demonstrated how to draw a regular octagon with 45° interior angles on the hyperbolic $(6, 6, 8)$ tessellation. The octagons in this paper are made of EVA foam Curvagons and can easily be joined along their edges to form a pair of pants.

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