Soccer Ball Torus: From Theory Crafting to Fibre Crafting

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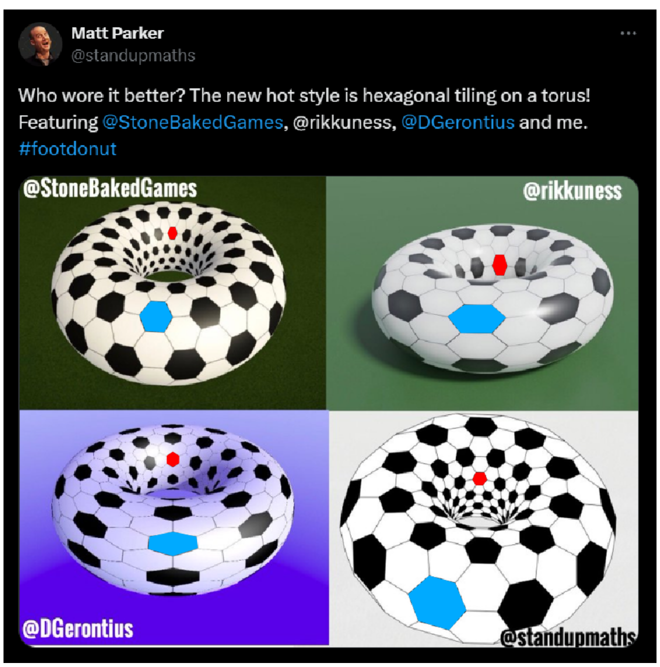
Abstract

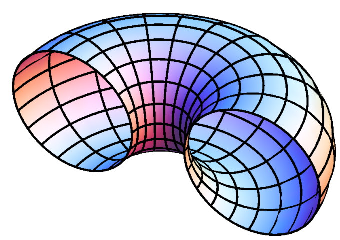
Inspired by Matt Parker's talk, we set out to crochet a hexagonal torus. It was an interesting journey filled with multidisciplinary challenges, ranging from learning how to do mathematical modeling, converting the results to papercrafts, figuring out how to create designs under constraints, and developing novel crochet techniques to overcome shaping challenges.

Motivation

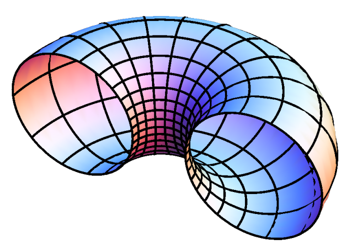
Spherical tilings have long captured the interest of the Bridges community. In 2024, Matt Parker presented a talk featuring a geometrically impossible soccer ball composed entirely of hexagons, showcasing a custom "impossiball" created by Jon-Paul Wheatley with hexagonal tiling on one side. He also briefly mentioned a torus tiled with hexagons (see Figure 1a), rendered in collaboration with his YouTube audience [1]. In this paper, we describe how to physically construct this curious object.



(a) *soccer ball torus tweet*



(b) *non-conformal map*



(c) *conformal map*

Figure 1: *Various parameterizations of a torus*

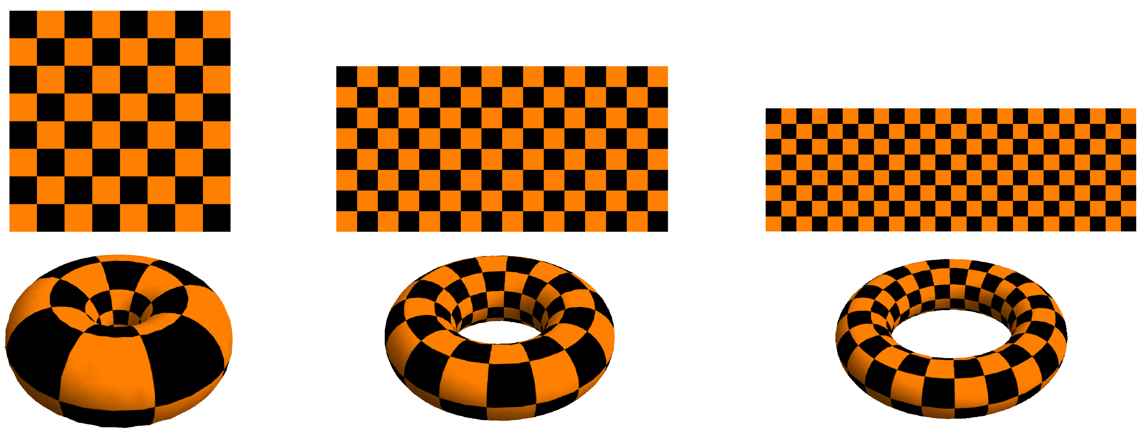
3D Modeling

The first step is to figure out how to tile the surface of a torus with hexagons. In Figure 1a, each torus has an outer hexagon coloured blue and an inner hexagon coloured red. Notice how the aspect ratios differ. The top-right torus has the largest discrepancy, with a vertically stretched red hexagon and a horizontally stretched blue hexagon. In contrast, the bottom-right torus has nearly regular hexagons everywhere. This latter aesthetic is achieved by using conformal maps, or locally angle-preserving functions. Figure 1b and Figure 1c show how conformal and non-conformal maps transform the shapes within a square grid.

Sullivan [2] describes a conformal map that maps a rectangular texture of width s and height 1 onto a torus. It can be generalized to take any rectangle of width m and n as input:

$$F_{m,n}(u,v) = \frac{\left(\frac{m}{n}\cos\frac{2\pi u}{m}, \frac{m}{n}\sin\frac{2\pi u}{m}, \sin\frac{2\pi v}{n}\right)}{\sqrt{\frac{m^2}{n^2}+1} - \cos\frac{2\pi v}{n}},$$

where $u \in [0,m]$, $v \in [0,n]$. Figure 2 illustrates how rectangles of different aspect ratios can conformally map onto different tori. When $m = n$, we have a square that forms a plump torus. The wider the aspect ratio, the thinner the torus becomes. Note that the rectangle cannot be taller than it is wide, or the resulting torus will be self-intersecting.



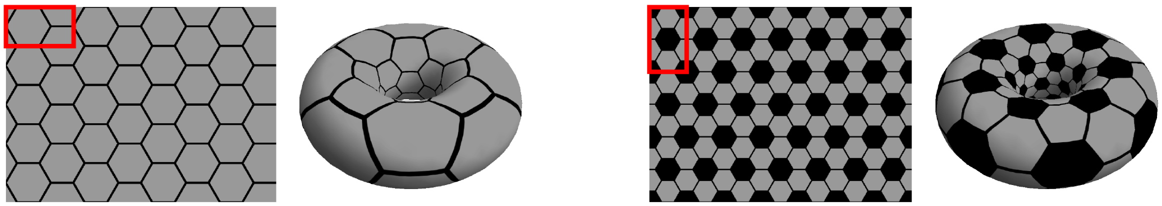
(a) $(m,n) = (8,8)$ **(b)** $(m,n) = (16,8)$ **(c)** $(m,n) = (24,8)$

Figure 2: *Square patterns conformally mapped onto tori*

To conformally map a rectangle onto a torus while preserving certain tiling patterns–such as a regular hexagonal tiling or a hexagonal tiling that supports a soccer ball pattern–the rectangle's aspect ratio must be compatible with the tiling's structure. Consider the regular hexagonal tiling, as shown in Figure 3a. The smallest repeating rectangular region—the *unit tile*, outlined in red—has an aspect ratio of $\sqrt{3}$ (width to height). Therefore, any rectangle that supports this tiling must have dimensions proportional to $(m,n) = (\sqrt{3}n_x, n_y)$, where n_x and n_y are the number of unit tiles along the horizontal and vertical directions. Now consider a hexagonal tiling that also supports a *soccer ball pattern* (see Figure 3b). The corresponding unit tile has an aspect ratio of $1/\sqrt{3}$. This implies that the rectangle must instead satisfy $(m,n) = (n_x, \sqrt{3}n_y)$ for some integers n_x and n_y.

From Rendering to Crafting

We could stop here if our goal were simply to render a soccer ball torus. But to actually craft it, we need to



(a) *Hexagonal tiling with* $(m, n) = (4\sqrt{3}, 5)$ **(b)** *Soccer ball tiling with* $(m, n) = (7, 3\sqrt{3})$

Figure 3: *Hexagonal patterns conformally mapped onto tori*

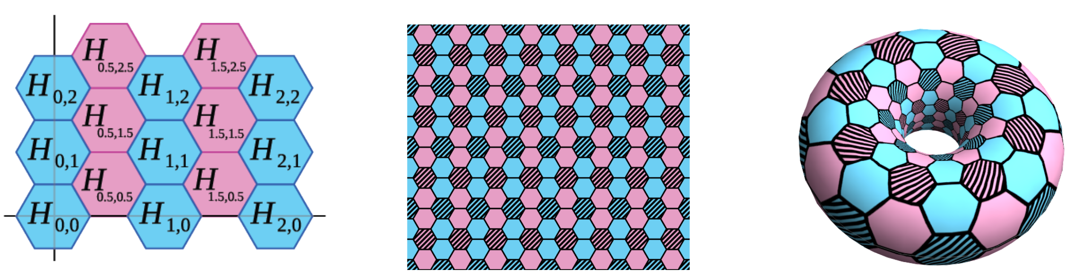
1. define all the hexagon vertices in the 2D texture,
2. map them conformally onto the torus, and
3. flatten the hexagons to get their exact dimensions.

First, we define a hexagonal tiling on $\mathbb{R}^2$ by letting $H_{i,j}$ denote a unit hexagon centered at $(3i, \sqrt{3}j)$. As shown in Figure 4a, the two sets $\{ H_{i,j} \mid i, j \in \mathbb{Z} \}$ (in blue) and $\{ H_{i+0.5,j+0.5} \mid i, j \in \mathbb{Z} \}$ (in pink) tile the plane together. This colour scheme will be used throughout the paper as a reminder of how the hexagons are laid out. We select the pattern from the centre of $H_{0,0}$ to the centre of $H_{8,12}$ because it supports a soccer ball pattern, illustrated in Figure 4b using hatching to indicate black polygons. This tiling has an aspect ratio of $m = 24$ and $n = 12\sqrt{3}$. Applying the conformal map $F_{m,n}$ transforms this planar tiling into one embedded on a torus (see Figure 4c). While this mapping changes the hexagon sizes, it preserves local angles.

Next we want to find the exact dimensions of these transformed hexagons by flattening onto a plane with a distance-preserving transformation. These flattened pieces can then be used to physically construct the torus. Strictly speaking, it's not possible to fully flatten the hexagons since the surface of a torus is not planar, but for tilings with enough hexagons, each tile is flat enough that we can assume they are planar.

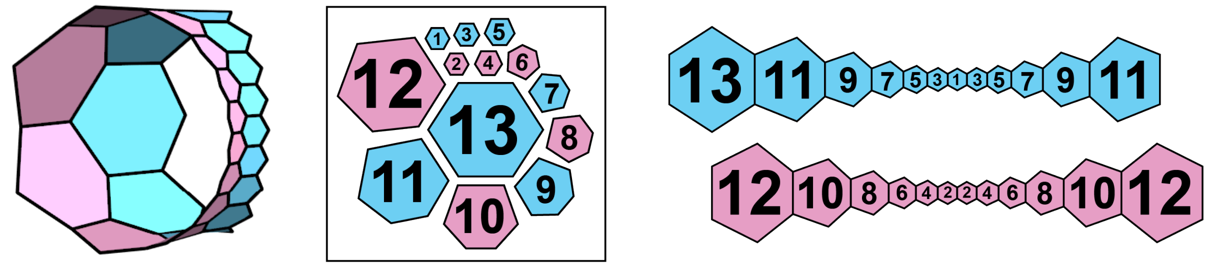
The tiled torus in Figure 4c is made up of 16 hexagon rings (8 blue and 8 pink). Figure 5a shows two such rings joined together. To flatten the hexagon rings into strips, first flatten each hexagon by orthogonally projecting it onto the plane spanned by two linearly independent edges. Label the flattened hexagons in order of increasing size, as shown in Figure 5b. Next, connect the blue, odd-numbered hexagons along their shared edges to form a strip, and do the same for the pink, even-numbered hexagons, as shown in Figure 5c.

To test our theory, we built several papercraft prototypes. The planar assumption may introduce some fitting issues, but paper provides a reliable proof of concept; if it assembles successfully with rigid material, it will perform even better with flexible crocheted fabric. Figure 6 shows a range of early prototypes exploring various sizes and patterns. We ultimately chose a design with 16 hexagonal strips and 13 distinct hexagon shapes.



(a) *Hexagon enumeration* **(b)** *Soccer tiling before* $F_{m,n}$ **(c)** *Soccer tiling after* $F_{m,n}$

Figure 4: *Hexagonal tiling before and after the conformal map* $F_{m,n}$ *is applied*



(a) *two hexagonal rings* **(b)** *individually flattened* **(c)** *rejoined hexagon strips*

Figure 5: *3D hexagon rings flattened and joined to form strips*

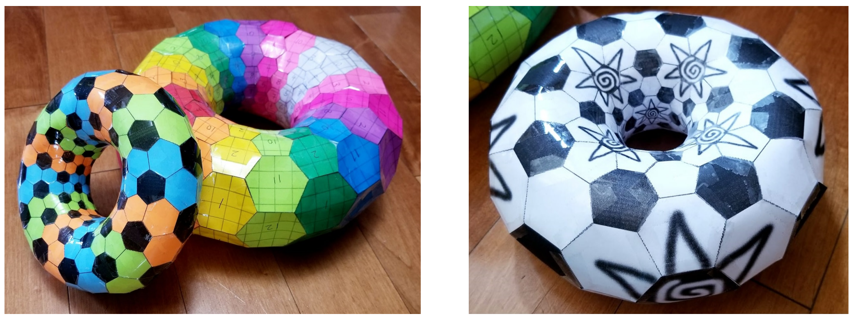


Figure 6: *Papercraft prototypes*

Basics of Crocheting

Having successfully created paper prototypes, we now shift our focus to crocheting the hexagons. To follow this section, some basic crochet concepts are helpful. Crochet creates fabric by interlocking loops of yarn, and unlike knitting—where stitch height is fixed—crochet allows for variable stitch heights, making it well suited for non-rectangular and three-dimensional forms. Figure 7 illustrates six stitches of increasing height.

These stitches can be combined to form regular hexagons. One approach is to crochet them flat (with back-and-forth rows), increasing or decreasing at the row ends to adjust the width (see Figure 8a). But more commonly, hexagons are crocheted radially, starting at the centre. The Granny hexagon in Figure 8b uses chains (shown as small circles) to emphasize corners while adding holes. An alternative from Raffamusa's blog [3] uses taller stitches to shape corners without creating holes.

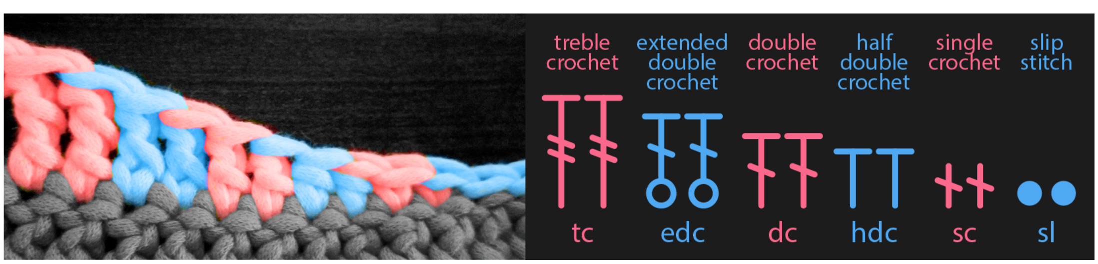
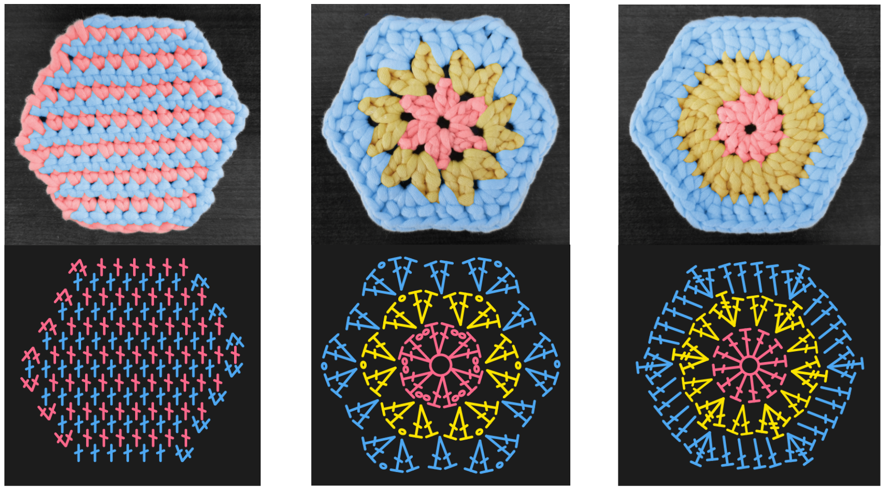


Figure 7: *Six common crochet stitch types*



(a) *Crocheted flat* **(b)** *With Granny stitches* **(c)** *With tall corner stitches*

Figure 8: *Different ways to crochet regular hexagons*

Crocheting Irregular Hexagons

The hexagons tiling the torus are irregular, since conformal maps preserve angles locally but not global lengths. To crochet these shapes, we avoid the flat method because it's not easily adapted to arbitrary angles and produces jagged edges. Given that the torus model requires stuffing to hold its shape, the Granny construction is also unsuitable because the large holes would let stuffing leak out. Instead, we adopt the tall corner stitch method. At this point, there are already several unanswered questions:

1. Where should the radial construction start, given the lack of a clear centre in an irregular hexagon?
2. How should stitch height vary across each round?
3. How should increases be distributed to match all six target side lengths?
4. How can we ensure the hexagon lies flat (i.e., has zero curvature)?

The initial plan was to measure individual stitches and treat the construction as a constraint problem. However, predicting the final shape is difficult, since stitches deform depending on their neighbours. A more practical approach is to print out each target hexagon and adjust its size or angles by trial and error.

Earlier, we selected a torus construction that consists of 13 distinct hexagons. Crocheting them all by trial and error would be daunting task, especially the larger ones where failure might only become apparent after 90% completion due to uneven construction. Ideally, we want a way to build up each hexagon gradually, reducing the risk of failure late in the process.

Fortunately, the hexagons can be constructed hierarchically. When the two strips of 13 hexagons (Figure 5c) are folded in a zigzag pattern, each hexagon nests within a larger one (see Figure 9). This mean we can start with, for example, Hexagon 1, and gradually build up to Hexagons 3, 5, 7, 9, 11, and 13 by crocheting a crescent-shaped border onto the previous hexagon. Similarly, Hexagon 12 can be build up from the even-numbered hexagons starting at Hexagon 2.

The crescent shapes are made out of triangles and trapezoids. The aforementioned tall corner stitch method doesn't work very well because it produces rounded corners (see Figure 10a). We made some modifications to improve the shaping. The first trick to creating sharper corners is to change the direction of the stitches. After the tall corner stitch, crochet the next few stitches directly on the body of the tall corner stitch. This forces an almost perpendicular turn, and as the stitches increase in height, a sharp corner is

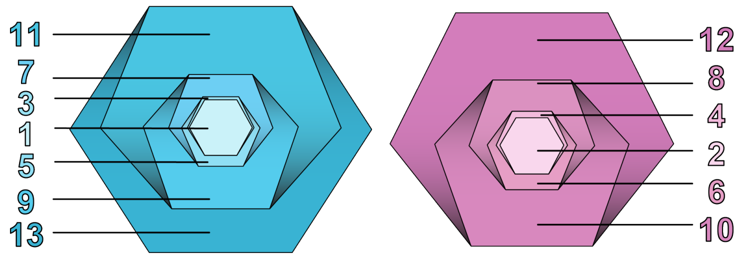
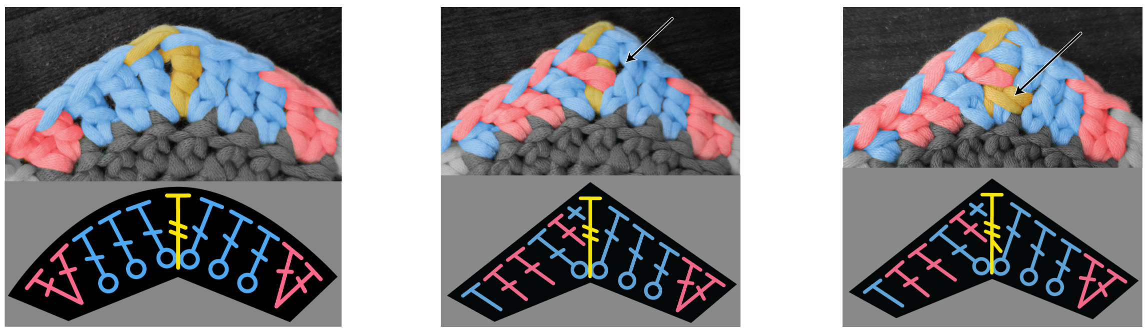


Figure 9: *Hierarchical relationships among odd-numbered hexagons and even-numbered hexagons*



(a) *Using a tall corner stitch creates a rounded corner*

(b) *Direction change sharpens the corner but leaves a hole*

(c) *Working the tall stitch onto the previous stitch closes the hole*

Figure 10: *How to fix the rounded corner issue without adding holes*

created (see Figure 10b). But having multiple stitches worked onto the taller stitch can pull the taller stitch sideways, leaving a small hole. The second trick is to close this hole by also working it onto the previous stitch, as if doing a decrease (see Figure 10c). This extra step pulls the taller stitch closer to the previous stitch in order to close the gap.

With new techniques and considerable patience, we were able to finalize the stitch patterns for all 13 hexagons. Figure 11 shows the pattern for the even-numbered hexagons in blue and the odd-numbered hexagons in pink. The corresponding hexagon labels are provided in Figure 9. This pattern is worked radially outward from the centre, starting with a magic circle. The beginning of each round is highlighted in a different colour and a small arrow indicates the crochet direction. Successive rounds are separated either by a slight background colour change or a white boundary. The colour change means you are still working on the same hexagon, while the white boundary indicates that you have completed a hexagon and is moving onto the next.

As an example, if you are making Hexagon 11 (the second largest blue hexagon), you will create a magic circle and work two rounds to complete Hexagon 1; work one round each for Hexagons 3, 5, 7, and 9; and work three more rounds to complete Hexagon 11 (the last two rounds in the pattern are skipped because they are for constructing Hexagon 13). Note that sometimes between rounds, you need to cut the yarn and start at a different location.

The pattern contains several unconventional notations we invented. When a stitch is meant to be worked on top of another stitch from two rounds previous, a dotted line is drawn between them to indicate alignment. When a stitch is drawn as if it's branching off of another stitch, it means the "branching" stitch should be worked onto the body of the "trunk" stitch. As discussed earlier, this branching helps create sharper corners.

A full version of the pattern, including step-by-step instructions for both the crocheting and the assembly, will be available on Ravelry, an online community and pattern library for knitters and crocheters.

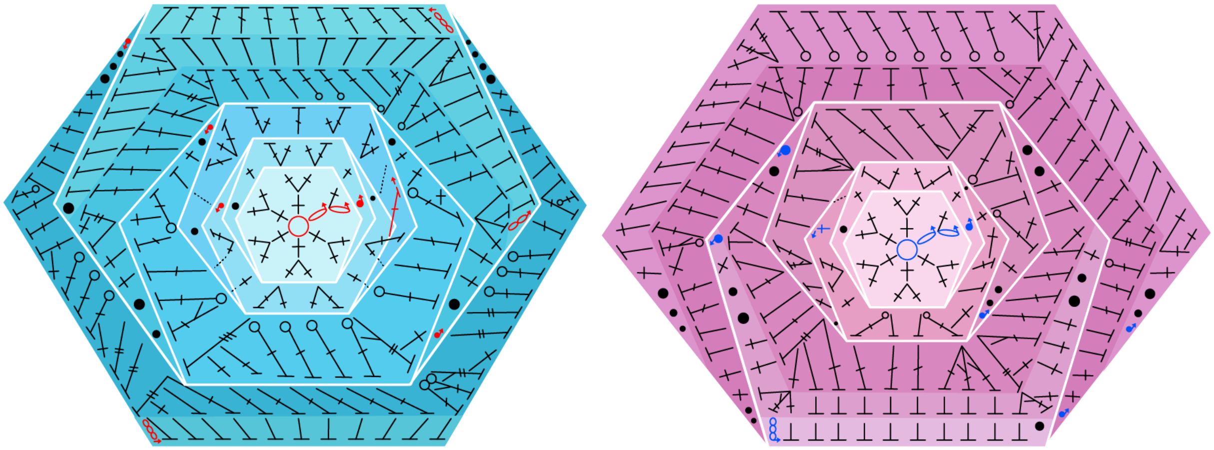


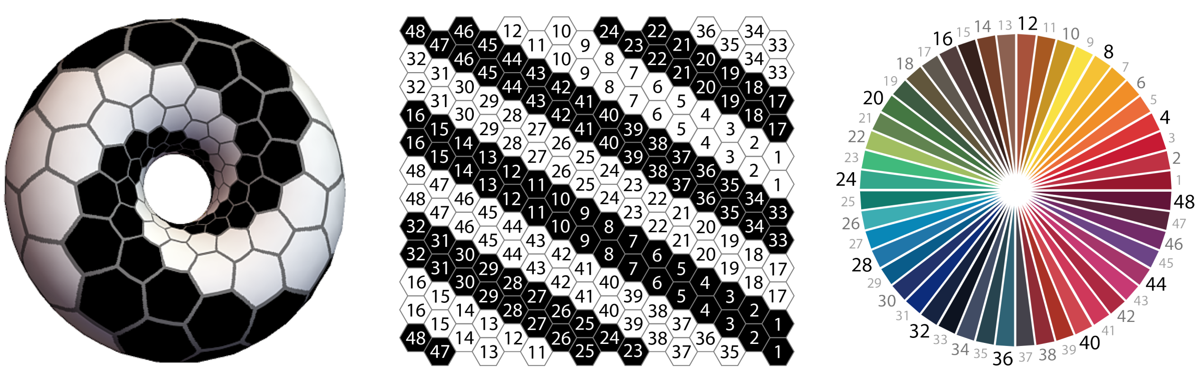
Figure 11: *Hierarchical stitch patterns for all 13 hexagonal tiles (draft and not to scale)*

Colour Design and Embellishments

Although the original soccer ball motif uses black and white, we opted for a more vibrant colour palette. To plan the layout, we began with the 2D hexagonal tiling (not yet mapped onto the torus) and drew thick diagonal lines. Due to the toroidal periodic boundary conditions, these diagonals loop back on themselves, splitting the tiling into two partitions: black and white (see Figure 12a). Each 2-thick diagonal contains 96 hexagons, which we label from 1 to 48 as shown in Figure 12b.

For colouring, we used the Schachenmayr Catania Amigurumi Set Box - Bright Colors, which includes 50 shades. After discarding black and white, we arranged the remaining 48 in a colour wheel to form a near-continuous gradient (Figure 12c). These colours were then mapped onto the numbered hexagons, resulting in the final colour scheme shown in Figure 13.

Although we completely overhauled the colour scheme, we still wanted to highlight the underlying symmetry of the soccer ball pattern. In Figure 13, dotted outlines indicate where the black hexagons would have been. Rather than colouring them black, we marked these hexagons with large safety eyes. Finally, since the diagonal bands from Figure 12a become less visible once coloured, we added metallic beaded borders to make them stand out.



(a) *two diagonal bands* **(b)** *colour layout* **(c)** *48 colours sorted*

Figure 12: *Colour scheme planning process*

Results

The project spanned several months, from mathematical modelling to the final crocheting. Hexagons were crocheted, assembled into strips, and sewn together. The entire piece was stuffed with polyfill to achieve its shape. The completed torus, *Ophanim*, is named after the celestial beings described as wheels covered in eyes. Figure 14 shows both sides of the torus, with colour gradients flowing through both diagonal bands.

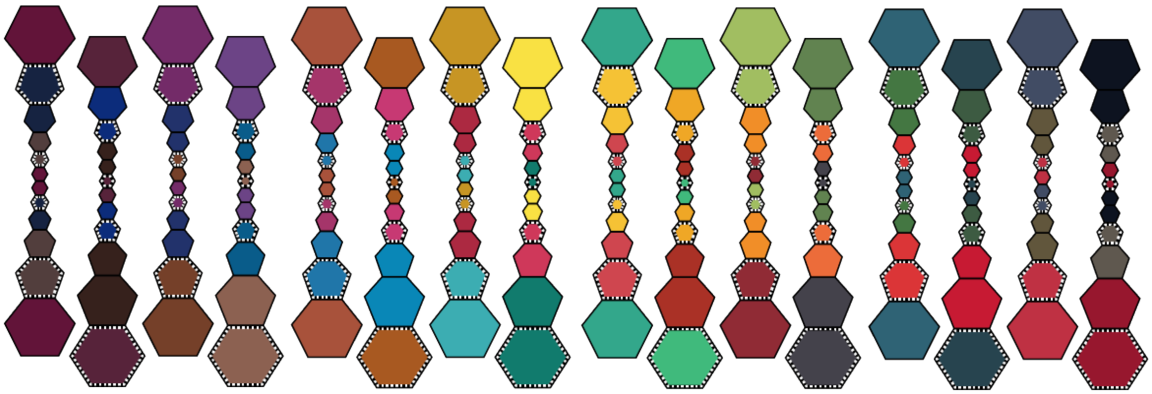


Figure 13: *Colour scheme applied to the hexagonal tiling*

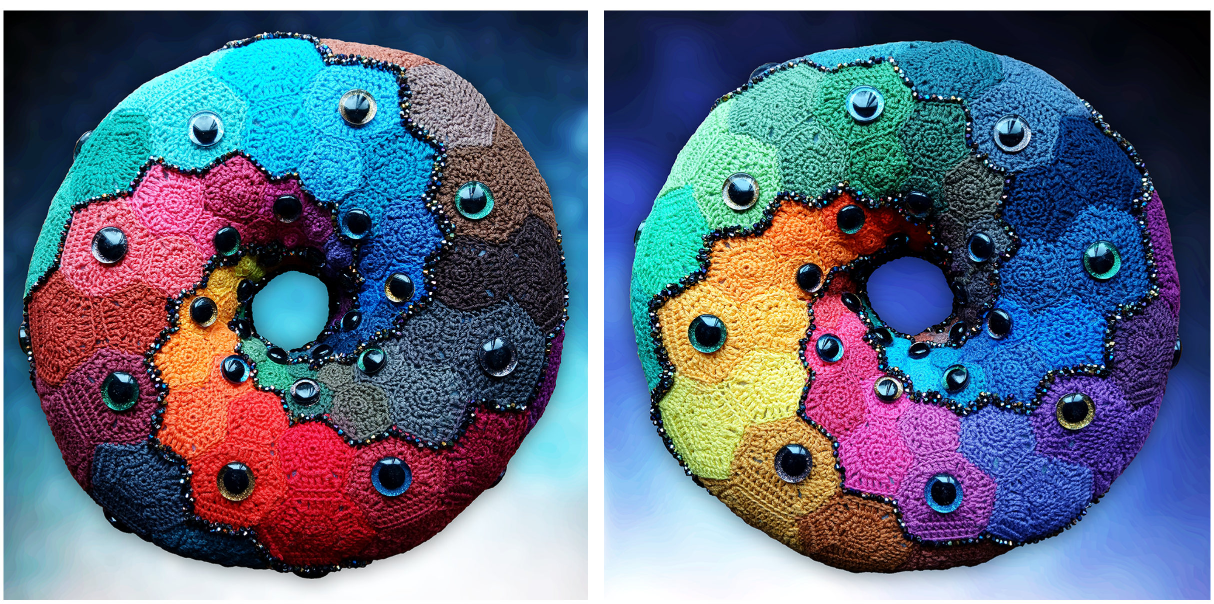


Figure 14: *Finished crocheted rainbow torus (40cm in diameter)*

References

[1] Matt Parker. “Who wore it better? The new hot style is hexagonal tiling on a torus!” *Twitter*, 2 July, 2018. https://x.com/standupmaths/status/1013834451166613509

[2] John M. Sullivan. “Conformal Tiling on a Torus” *Bridges Conference Proceedings*, Coimbra, Portugal, Jul. 27–31, 2011, pp. 593–596. http://archive.bridgesmathart.org/2011/bridges2011-593.pdf.

[3] Raffaella Tassoni. “How to crochet a completely solid hexagons –without gaps!”, *Raffamusa Designs*. https://raffamusadesigns.com/crochet-solid-hexagon-no-gaps/.