An Initial Attempt at a Mathematical Treatment of Translational Coordinate-Motion Puzzles

Supplement

George I. Bell

Louisville, Colorado, USA; gibell@comcast.net

Rods and Holes Procedure for a Regular Polygon

Here we apply the rods and holes procedure to a regular polygon with *n* sides. The 3rd dimension plays no role here as no movement is possible in *z*. Therefore, we drop the *z*-coordinate of all vectors.

The piece shape is a triangular sector of angle $2\pi/n$, as shown in Figure 1s. We can think of the rod as infinitely thin, i.e. a line which must be aligned with another line (the hole) in the next piece.

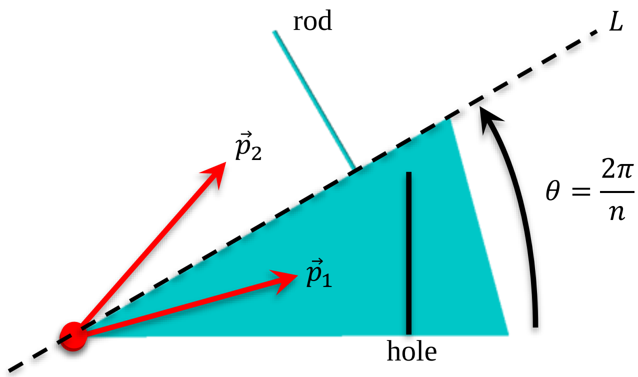

Figure 1s: *Piece 1, a triangular sector of angle* $2\pi/n$ *with rod and hole represented by lines.*

We write $\vec{p}_i = (p_i, q_i)$ and consider that $\vec{p}_{n+1}$ is the same as $\vec{p}_1$. As pieces 1 and 2 move apart, the condition that the rod from piece 1 slides smoothly along the hole of piece 2 is $\vec{p}_2 - \vec{p}_1 \perp L$, where L is the line separating pieces 1 and 2. We can write this condition for general *i* as

$$(\vec{p}_{i+1} - \vec{p}_i) \cdot \left(cos\frac{2\pi i}{n}, \sin\frac{2\pi i}{n} \right) = 0, i = 1,2, \cdots, n. \qquad (1s)$$

Which can be written as a set of *n* linear equations for the unknowns p_i and q_i,

$$(p_{i+1} - p_i)\cos\frac{2\pi i}{n} + (q_i - q_{i+1})\sin\frac{2\pi i}{n} = 0 \qquad (2s)$$

In addition we remove any overall translation by requiring

$$\sum_{i=1}^{n} p_i = 0 \qquad (3s)$$

$$\sum_{i=1}^{n} q_i = 0 \qquad (4s)$$

Equations (2s,3s,4s) constitute a set of $n + 2$ linear homogeneous equations with $2n$ unknowns. Such a system of equations has a null space of dimension $2n - (n + 2) = n - 2$. This is the number of degrees of freedom in this mechanical system (when rotations are not allowed).

For example, when $n = 5$ we can use a numerical matrix package to obtain a basis for the null space:

```
N[NullSpace[m5]]

{{0., 2., 3.3022, 0.927051, -0.224514, -3.92705, -3.07768, 0., 0., 1.},
 {1., -1.6246, -2.45492, -0.502029, -0.545085, 2.12663, 1., 0., 1., 0.},
 {0., -3.23607, -3.80423, -2., 0.726543, 4.23607, 3.07768, 1., 0., 0.}}
```

Each row gives a null space basis vector of the form $\{p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4, p_5, q_5\}$. Figure 2s shows the third basis vector.

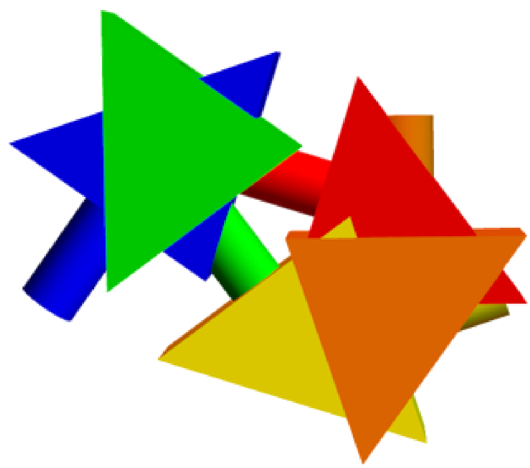

Figure 2s: *The third basis vector showing piece movements. The orange piece (#5) is stationary.*

The Piece Movement Vectors (PMVs) also satisfy the linear Equations (2s,3s,4s), but we add additional constraints that the pieces cannot overlap. These constraints are:

$$(\vec{p}_{i+1} - \vec{p}_i) \cdot \left(-\sin\frac{2\pi i}{n}, \cos\frac{2\pi i}{n}\right) \geq 0, i = 1,2,\cdots,n.$$

or

$$(p_i - p_{i+1})\sin\frac{2\pi i}{n} + (q_{i+1} - q_i)\cos\frac{2\pi i}{n} \geq 0 \qquad (5s)$$

We note that neither of the three basis vectors for the null space satisfy this constraint.

The solution to Equations (2s,3s,4s) under the n constraints (5s) is not a vector space, but rather a convex cone. It is more difficult to calculate the PMVs, but using a mechanical model of the $n = 5$ puzzle the solution in Figure 3s was discovered. The exact form of this solution vector, in the same form as above, is:

$$\{\delta, 1, \delta, 1, -4\delta, 0, \delta, -1, \delta, -1\} \; where \; \delta = -\frac{1}{5}\tan\frac{4\pi}{5} \cong 0.1453$$

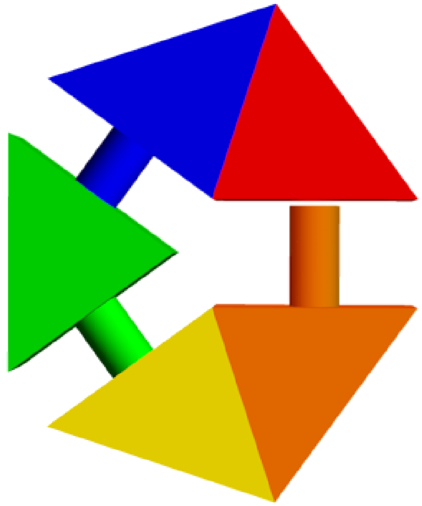

Figure 3s: *A PMV for the n=5 case.*

The PMV in Figure 3s can be rotated to make four more PMV's. Each is unique in the sense that it cannot be written as a non-negative linear combination of the other 4. Thus, I claim that the case $n = 5$ has 3 degrees of freedom, and disassembly dimension $D \geq 5$.