An Initial Attempt at a Mathematical Treatment of Translational Coordinate-Motion Puzzles

Supplement

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Rods and Holes Procedure for a Regular Polygon

Here we apply the rods and holes procedure to a regular polygon with $n$ sides. The 3rd dimension plays no role here as no movement is possible in $z$. Therefore, we drop the $z$-coordinate of all vectors.

The piece shape is a triangular sector of angle $2\pi/n$, as shown in Figure 1s. We can think of the rod as infinitely thin, i.e. a line which must be aligned with another line (the hole) in the next piece.

**Figure 1s: Piece 1, a triangular sector of angle $2\pi/n$ with rod and hole represented by lines.**

We write $\vec{p}_i = (p_i, q_i)$ and consider that $\vec{p}_{n+1}$ is the same as $\vec{p}_1$. As pieces 1 and 2 move apart, the condition that the rod from piece 1 slides smoothly along the hole of piece 2 is $\vec{p}_2 - \vec{p}_1 \perp L$, where $L$ is the line separating pieces 1 and 2. We can write this condition for general $i$ as

$$(\vec{p}_{i+1} - \vec{p}_i) \cdot \left( \cos \frac{2\pi i}{n}, \sin \frac{2\pi i}{n} \right) = 0, \ i = 1, 2, \ldots, n. \quad (1s)$$

Which can be written as a set of $n$ linear equations for the unknowns $p_i$ and $q_i$,

$$(p_{i+1} - p_i) \cos \frac{2\pi i}{n} + (q_i - q_{i+1}) \sin \frac{2\pi i}{n} = 0 \quad (2s)$$

In addition we remove any overall translation by requiring

$$\sum_{i=1}^{n} p_i = 0 \quad (3s)$$

$$\sum_{i=1}^{n} q_i = 0 \quad (4s)$$
Equations (2s,3s,4s) constitute a set of \( n + 2 \) linear homogeneous equations with \( 2n \) unknowns. Such a system of equations has a null space of dimension \( 2n - (n + 2) = n - 2 \). This is the number of degrees of freedom in this mechanical system (when rotations are not allowed).

For example, when \( n = 5 \) we can use a numerical matrix package to obtain a basis for the null space:

\[
\begin{align*}
N[&\text{NullSpace[m5]}] \\
&\left\{ \{0., 2., 3.3022, 0.927051, -0.224514, -3.92705, -3.07768, 0., 0., 1. \}, \\
&\{1., -1.6246, -2.45492, -0.502029, -0.545085, 2.12663, 1., 0., 1., 0. \}, \\
&\{0., -3.23607, -3.80423, -2., 0.726543, 4.23607, 3.07768, 1., 0., 0. \} \}
\end{align*}
\]

Each row gives a null space basis vector of the form \( \{p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4, p_5, q_5\} \). Figure 2s shows the third basis vector.

**Figure 2s:** The third basis vector showing piece movements. The orange piece (#5) is stationary.

The Piece Movement Vectors (PMVs) also satisfy the linear Equations (2s,3s,4s), but we add additional constraints that the pieces cannot overlap. These constraints are:

\[
(\vec{p}_{i+1} - \vec{p}_i) \cdot \left( -\sin \frac{2\pi i}{n}, \cos \frac{2\pi i}{n} \right) \geq 0, \quad i = 1, 2, \ldots, n.
\]

or

\[
(p_i - p_{i+1}) \sin \frac{2\pi i}{n} + (q_{i+1} - q_i) \cos \frac{2\pi i}{n} \geq 0 \quad (5s)
\]

We note that neither of the three basis vectors for the null space satisfy this constraint.

The solution to Equations (2s,3s,4s) under the \( n \) constraints (5s) is not a vector space, but rather a convex cone. It is more difficult to calculate the PMVs, but using a mechanical model of the \( n = 5 \) puzzle the solution in Figure 3s was discovered. The exact form of this solution vector, in the same form as above, is:

\[
\{\delta, 1, \delta, 1, -4\delta, 0, \delta, -1, \delta, -1\}\text{ where } \delta = -\frac{1}{5} \tan \frac{4\pi}{5} \approx 0.1453
\]
The PMV in Figure 3s can be rotated to make four more PMV’s. Each is unique in the sense that it cannot be written as a non-negative linear combination of the other 4. Thus, I claim that the case $n = 5$ has 3 degrees of freedom, and disassembly dimension $D \geq 5$. 

Figure 3s: A PMV for the $n=5$ case.