

Using Kronecker Powers to Create Bead Art

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Abstract

In this workshop, we'll explore how to create visual patterns using Kronecker powers on square matrices. Participants will create unique designs using plastic beads. We'll also discuss pedagogical strategies to encourage participants to pose and explore original problems about designs created using this method. We will share Python code that will allow for participants to further explore these patterns.

Introduction

As mathematics teachers, we aim to foster a sense of curiosity in our students. Creating mathematical art encourages students to take risks and ask “what-if” questions. The activity presented below is accessible to students in middle and high school. This activity creates bead art using a matrix operation that may be new to participants. Creating art using this matrix operation presents many opportunities to tweak parameters and explore the outcomes of the change, allowing opportunities for creativity.

Mathematical Background

Let's begin by considering the sequence of three matrices shown in Figure 1:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 1: A sequence of three matrices.

Examining these matrices closely, you may discover a pattern in the way that A_2 and A_3 were constructed from A_1 .

The matrix operation used to create this sequence is known as the *Kronecker product*. The Kronecker product \otimes is defined as follows, where a_{ij} refers to the entry in the i^{th} row and the j^{th} column of A :

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then $A \otimes B$ will be a $mp \times nq$ matrix. Referring back to Figure 1, we can see that:

$$A_2 = A_1 \otimes A_1 = \begin{bmatrix} 1 \cdot A_1 & 1 \cdot A_1 \\ 1 \cdot A_1 & 0 \cdot A_1 \end{bmatrix}$$

and

$$A_3 = A_1 \otimes A_2 = \begin{bmatrix} 1 \cdot A_2 & 1 \cdot A_2 \\ 1 \cdot A_2 & 0 \cdot A_2 \end{bmatrix}$$

We could continue to create terms in this sequence by taking further *Kronecker powers* of A_1 . The n th Kronecker power $A^{\otimes n}$ is defined as taking the Kronecker product of A with itself n times:

$$A^{\otimes n} = \underbrace{A \otimes A \otimes A \dots \otimes A}_{n \text{ times}}$$

Thus, another way to write the matrices shown in Figure 1 is :

$$A_2 = A_1^{\otimes 2} \text{ and } A_3 = A_1^{\otimes 3}.$$

It is interesting to take Kronecker powers of matrices whose entries are all 0s or 1s, the set of (0,1)-matrices. We find Kronecker powers of these matrices interesting for two reasons.

First, they are *closed* under the operation of the Kronecker product. If we take the Kronecker product of two (0,1)-matrices, the result is also a (0,1)-matrix.

Secondly, Kronecker powers of (0,1)-matrices are *self-similar*. By this, we mean that smaller parts of the resulting matrix will resemble the structure of the overall matrix. We will see that this property will result in visualizations that approximate fractals that participants may already be familiar with.

A (0,1)-matrix is equivalent to a grid where each cell can be colored black or white. We'll use this equivalence in the next section to create beaded pieces of art.

Creating Bead Art

Now we will use Kronecker powers to create physical objects using plastic melty beads. The beads provide an easy entry point for visual exploration. In the United States, the beads we're using can be bought inexpensively in the thousands under the brand name "Perler Beads." They are also known as Hama beads or Nabbi beads in other countries.

To complete the following activity, each participant will need a plastic pegboard that the beads fit onto and at least 200 beads of each color. A pair of tweezers is also helpful in placing the beads. Parchment paper and an iron will also be needed for melting the beads together.

The first step is for the participant to create their seed (0,1)-matrix. We recommend beginning with a 2×2 seed matrix. While this suggestion restricts us to only a few starting matrices, it is possible to be creative with larger seed matrices after exploring the underlying structure of these matrices using Kronecker powers and beads. For the examples in this paper, we'll refer to the seed matrix as matrix A . For the following example, the seed matrix A will be the matrix A_1 from Figure 1.

The second step is to decide which color will represent 0 and which color will represent 1. We'll represent 1 with black beads and 0 with white beads.

The next step is to put one copy of beads that represent the seed matrix onto the pegboard. It should be placed near the corner of the board. It will not be part of the finished design but will serve as a reminder of the seed matrix. This setup is shown in Figure 2.



Figure 2: *The initial setup with the seed matrix.*

Now we create a representation of $A^{\otimes 2}$. To do so, we want to think of creating a 4×4 matrix out of four 2×2 matrices. Each of the 2×2 matrices will be a quadrant of the 4×4 matrix. To decide how to create each of these quadrants, we'll look at the seed matrix. If the bead in the corresponding quadrant is a 1, then a copy of the seed matrix is built in that quadrant. If the bead in the corresponding quadrant is a 0, then the entire quadrant is built out of white beads. Figure 3 shows the process after this step.

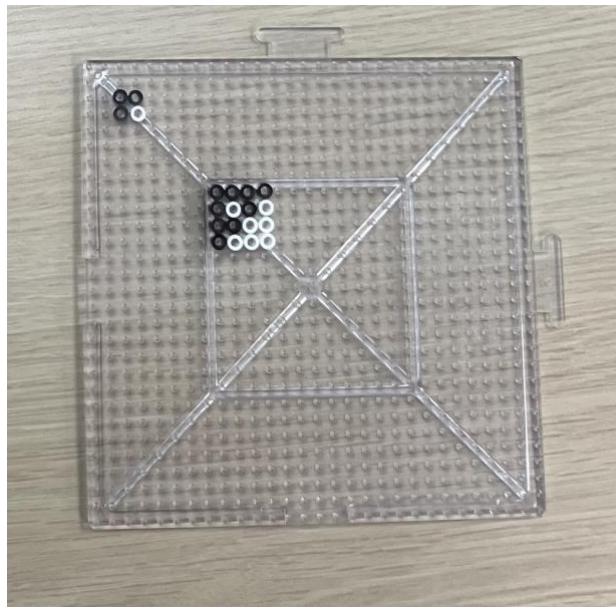


Figure 3: *The seed matrix and $A^{\otimes 2}$.*

We continue similarly to create the visual representation $A^{\otimes 3}$. A copy of $A^{\otimes 2}$ is now placed in any quadrant that has a 1 in the corresponding position of the seed matrix. White beads fill in the entire quadrant if a 0 is in the corresponding position of the seed matrix. Figure 4 shows $A^{\otimes 3}$ and $A^{\otimes 4}$.

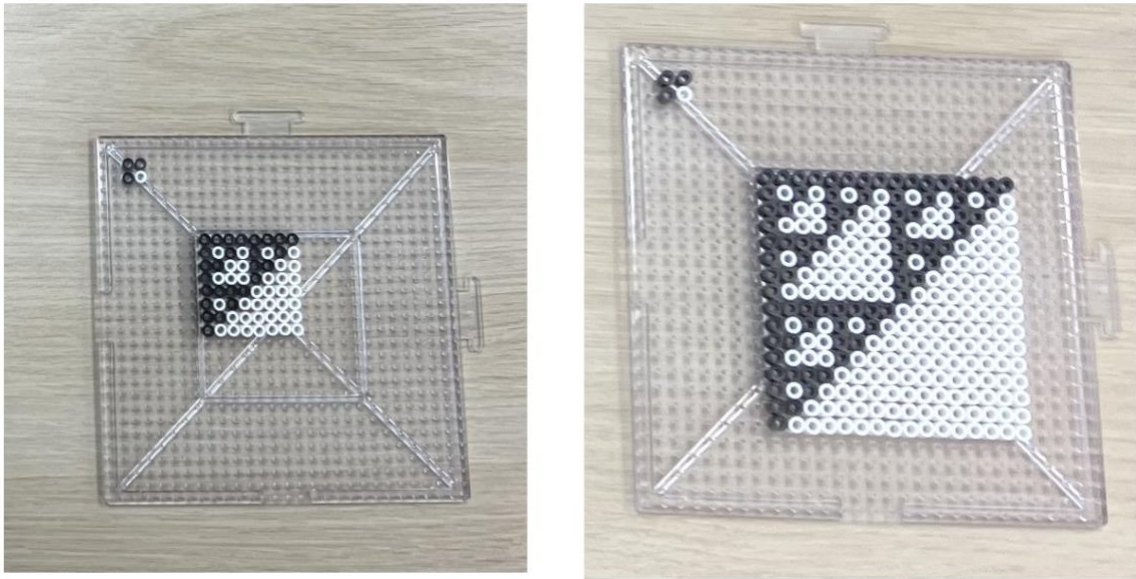
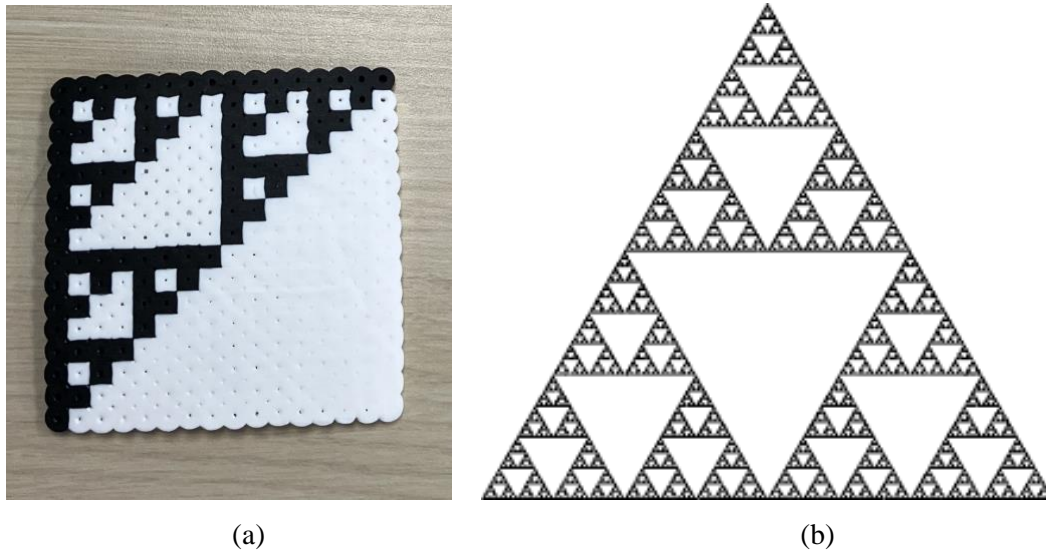


Figure 4: $A^{\otimes 3}$ and $A^{\otimes 4}$.

Once all of these beads are placed, the copy of the seed matrix that was placed in the corner can carefully be removed and the beads can be fused. To fuse, the beads should be covered with a piece of parchment paper and then an iron can be placed gently on top of the parchment paper until the beads are fully fused. This process generally takes about 10 – 30 seconds. Once the first side has been fused, the beads can carefully be removed from the pegboard and turned over. Then the parchment paper should be placed on top of the other side of the beads and the iron should be used to fuse that side. The example that we've created approximates the Sierpinski Triangle. Figure 5 shows the final product after the beads have been fused next to the Sierpinski Triangle.



(a) (b)
Figure 5: (a) final product, (b) Sierpinski Triangle.

Starting with a seed matrix with larger dimensions allows for many more starting possibilities. Figures 6 and 7 show the starting seed matrix, the beaded version of $A^{\otimes 3}$, and a picture of a well-known fractal the beaded version approximates.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

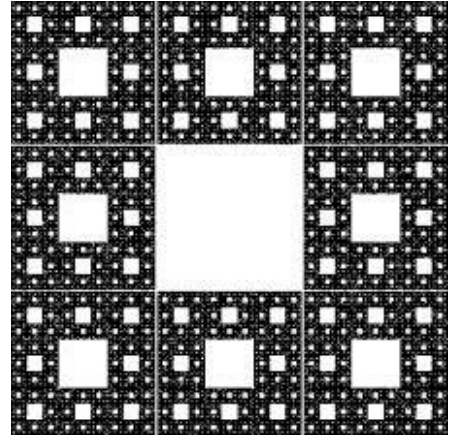
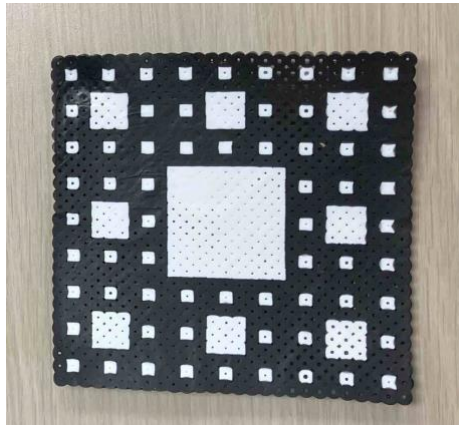


Figure 6: (a) seed matrix, (b) beaded result, (c) Sierpinski carpet.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

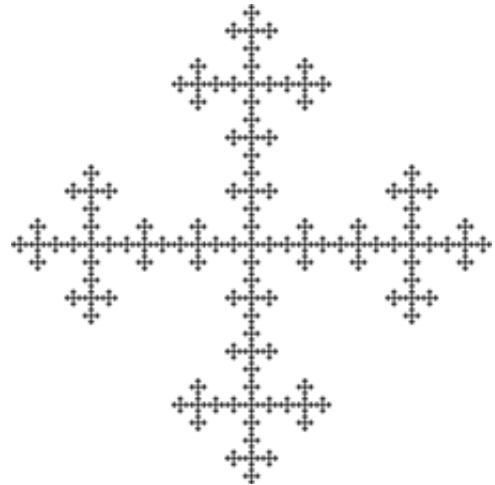
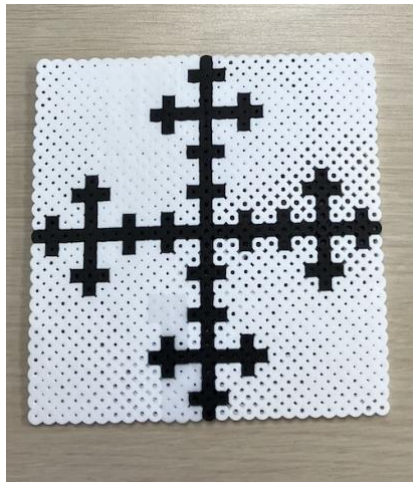


Figure 7: (a) seed matrix, (b) beaded result, (c) Vicsek fractal

Pedagogical Implications and Extensions

Jeremy Kilpatrick, a prolific researcher of mathematics education stated, “[T]he experience of discovering and creating one’s own mathematics problems ought to be part of every student’s education” [2]. He argues that almost all of the problems that students encounter *outside* of the classroom will not be given to them by a textbook or authority figure, but rather ones that they must formulate themselves. One way that teachers can help facilitate students creating their own problems is to help students make a new problem out of an existing one. This generation of new problems by students is called *problem posing*.

Brown and Walter discuss what they call the “What-If-Not” strategy for problem posing [1]. This strategy consists of making a list of attributes about a problem that already exists and exploring how one change modifies the problem. Designs created by Kronecker powers allows the practice of this method of problem generation. The following properties can be listed about the process we’ve explored:

1. The seed matrix is square.
2. The seed matrix only has entries of 0 and 1.
3. Kronecker powers are used to generate a new matrix.

Via problem-posing, it is possible to explore how modifying these parameters can change the behavior. These new explorations help unlock understanding about underlying structure and patterns. Here are a few examples of possible problems that could be explored:

1. The seed matrix is a non-square matrix, such as a 2×3 matrix.
2. The seed matrix is not a $(0, 1)$ -matrix. For example, the seed matrix could be a $(-1, 1)$ -matrix.
3. The seed matrix has entries with absolute values of less than 1. A grayscale or color spectrum is used to color the resulting values.

Figures 8 and 9 show a possible outcome of exploring using a $(-1, 1)$ -matrix as a seed matrix. While these matrices do not have the same self-similar property that the $(0, 1)$ -matrices exhibited, we are still intrigued by the designs they create. One possible question that could be explored are the necessary conditions for Kronecker powers to result in a self-similar matrix.

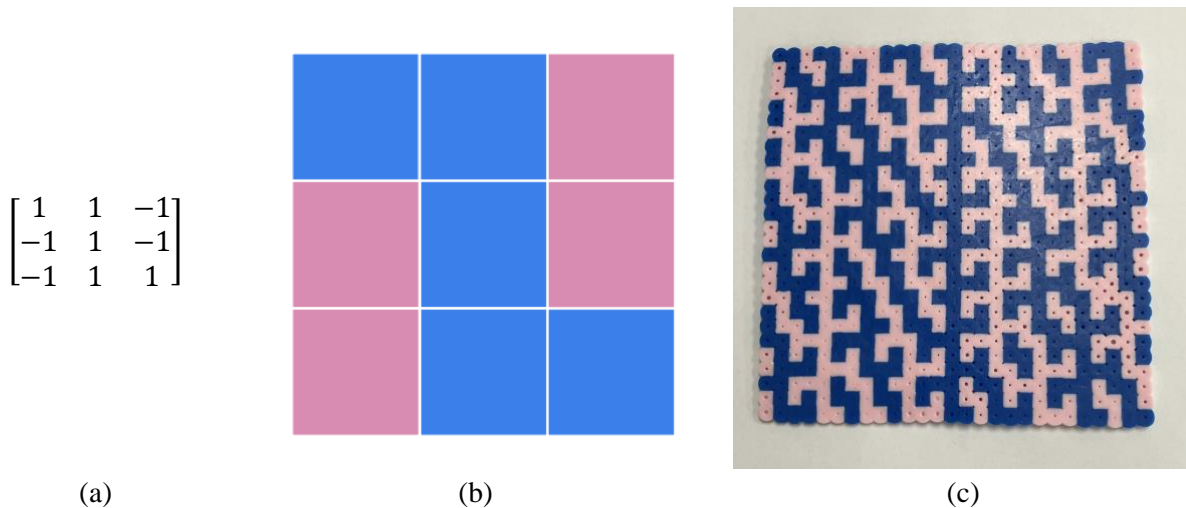
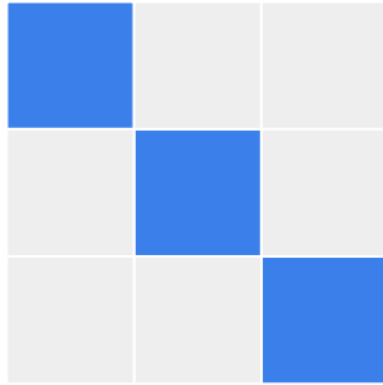


Figure 8: (a) seed matrix, (b) visualization of the seed matrix, (c) beaded result of Kronecker powers

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$



(a)

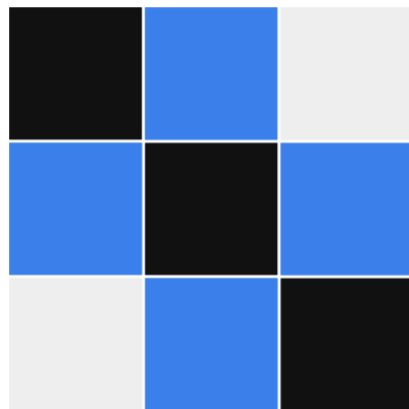
(b)

(c)

Figure 9: (a) seed matrix, (b) visualization of the seed matrix, (c) beaded result of Kronecker powers

Figure 10 shows one example of a visualization that can be made if the seed matrix is a $(-1,0,1)$ -matrix.

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$



(a)

(b)

(c)

Figure 10: (a) seed matrix, (b) visualization of the seed matrix, (c) beaded result of Kronecker powers

In addition to creating a new mathematical problem, it is possible to explore other constraints such as the ratio of black to white squares as we take $\lim_{n \rightarrow \infty} A^{\otimes n}$.

Exploring Kronecker Powers with Python Code

While creating the matrices out of beads provides a fruitful tactile experience to make a physical object, using beads to explore new problems may be time-consuming and impractical. As an option to support problem posing exploration, it is possible to use Python code to help automate the process of building matrices through Kronecker powers [3].

The Python library `numpy` has the `kron` operation. The following code segment can be used to define and print a new matrix as the Kronecker product of `h0` with itself. This code quickly generates the example shown earlier in Figure 1:

```
h0 = [[1,1], [1,0]]
h1 = kron(h0,h0)
h2 = kron(h1,h0)
print(h2)

[[1 1 1 1 1 1 1 1]
 [1 0 1 0 1 0 1 0]
 [1 1 0 0 1 1 0 0]
 [1 0 0 0 1 0 0 0]
 [1 1 1 1 0 0 0 0]
 [1 0 1 0 0 0 0 0]
 [1 1 0 0 0 0 0 0]
 [1 0 0 0 0 0 0 0]]
```

The script in [3] can be used to create a visual display of the entries in a matrix, where 1 corresponds to black and 0 corresponds to white. The function that displays the contents of a matrix is named `display_matrix`. The script can be used to investigate other modifications, such as using entries besides 1 and 0, or using a non-square seed matrix.

Summary and Conclusions

Using matrices to generate mathematical visualization is accessible for all. For students, even if students are unfamiliar with matrices or matrix theory is not in the curriculum, generating new questions and exploring their outcomes are powerful mathematical dispositions. This activity bridges the tactile experience of making physical art with using digital tools like Python to create more advanced patterns.

Acknowledgements

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References

- [1] S. Brown and M. Walter. *The Art of Problem Posing*. Lawrence Erlbaum Associates, Inc., Publishers 1990.
- [2] J. Kilpatrick. "Problem formulating: Where do good problems come from." *Cognitive science and mathematics education*, 123–147. 1987.
- [3] <https://tinyurl.com/KroneckerArtProblemPose>