Serialism Applied to a Mathematical Curiosity: The Musical Analogue to the Smallest Known Sierpiński Number

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Abstract

A *Sierpiński number* is a positive odd integer k such that $k \cdot 2^n + 1$ is composite for all positive integers n. *Serialism* is a compositional technique wherein various elements of a musical piece (pitches, rhythms, dynamics, etc.) are determined by a central repeating, ordered pattern. By formalizing auditory analogues to the mathematical concepts and information encoded in Sierpiński numbers, we produce a direct musical analogue to the smallest known Sierpiński number, as well as a methodology for musically interpreting similar mathematical objects. Further, using this source material, we present a serial composition.

Introduction

To contextualize our final result, we proceed in three sections. In Section 1, we give a brief history and description of Sierpiński numbers. In Section 2, we cover the musical preliminaries. In Section 3, we derive the auditory analogue to a Sierpiński number and present a serial composition.

The Smallest Known Sierpiński Number

A Sierpiński number is a positive odd integer k such that $k \cdot 2^n + 1$ is composite for all positive integers n. Sierpiński numbers are a curiosity of the integers. Upon first considering the premise, it is not obvious if these numbers exist, or if, heuristically speaking, they *should* or *should not* exist at all. To convince the reader of the novelty of this property, consider when $k = 7$. The numbers of the form $k \cdot 2^n + 1$ as n varies among the positive integers include

$$\underbrace{7 \cdot 2^1 + 1}_{15}, \qquad \underbrace{7 \cdot 2^2 + 1}_{29 \text{ (prime)}}, \qquad \underbrace{7 \cdot 2^3 + 1}_{57}, \qquad \ldots$$

and so on. Our choice of $k = 7$ fails miserably at being a Sierpiński number; we hit our first prime when $n = 2$. The desired property is not trivial, and, in fact, is exceedingly rare among the integers. Further, Sierpiński numbers *cannot* be found through brute force, since there are infinitely many cases to consider. So, how does one come upon such a number?

Historically, a Sierpiński number is found using a *covering*, which we will define in Section 3. The argument assures us that $k \cdot 2^n + 1$ is composite by being divisible by at least one of finitely many fixed primes less than $k \cdot 2^n + 1$ *for any choice of* n. In other words, the finite collection of primes is constant as n roams over the positive integers. This collection of primes is referred to as the *prime covering set*. The interested reader will find a rigorous construction of Sierpiński numbers in [1]. For our purposes, we highlight that a Sierpiński number is determined by two pieces of information — a *covering* and its corresponding *prime covering set*. These are the mathematical objects that we will derive musical analogues for in Section 3.

The first Sierpiński number was found by W. Sierpiński in 1960 (see [9]). His covering produced $k = 5511380746462593381$. As one may guess, Sierpiński numbers tend to be rather large. The smallest known Sierpiński number is 78557. It was found by J. Selfridge (unpublished) in 1962. Whether or not 78557 is truly the smallest, commonly refered to as "*The Sierpiński Problem*," is the greatest open question in this area. To address this question, in 2002, a collective of academics started the project *Seventeen or Bust*. As the name suggests, all but seventeen of the positive integers less than 78557 had been shown to *not* be Sierpiński numbers at the time of the project's inception (see [2] for further detail). With this in mind, our work is not only an academic and creative pursuit, but also an homage to this special number, and all of the work that has gone into *Seventeen or Bust*. Before addressing the musical preliminaries, we restate the key points. The smallest known Sierpiński number is 78557. There is a unique *covering* and a unique *prime covering set* associated with 78557.

Musical Preliminaries

The purpose of this section is to provide the appropriate musical background to justify and contextualize our musical interpretations and choices detailed in Section 3. We will highlight the key points as we proceed. We begin with a brief discussion of the *notes* (or *pitches*) of modern Western music. Recall there are twelve notes per octave. They are ordered C, $C^\sharp$ (or $D^\flat$), D, $D^\sharp$ (or $E^\flat$), E, F, $F^\sharp$ (or $G^\flat$), G, $G^\sharp$ (or $A^\flat$), A, $A^\sharp$ (or $B^\flat$), and B. By convention, octave 1 begins at the approximate frequency of 32.70 Hertz (Hz), and we refer to this pitch as $C1$. A doubling of this frequency begins octave 2, and we refer to this pitch as $C2$. As seen in these examples, one may indicate the octave a note falls under with the appropriate integer just after the note name. We will use the term *pitch class* to refer to a note's letter name, irrespective of octave.

The methodology and reasoning behind the placement of the remaining 11 notes (in terms of frequency) between $C1$ and $C2$ falls under the purview of *tuning systems*. Under the tuning system of *just intonation*, the frequencies are chosen based on particular integer ratios. In Table 1, we present several musical intervals that occur within an octave and the ratio characterizing the interval under just intonation.

Table 1: *The Just Intonation Whole Number Ratios of Musical Intervals.*

Interval	Example	Ratio
Unison	$C1 \to C1$	1 : 1
Minor Third	$C1 \to E^\flat 1$	5 : 6
Major Third	$C1 \to E1$	4 : 5
Perfect Fourth	$C1 \to F1$	3 : 4
Perfect Fifth	$C1 \to G1$	2 : 3
Minor Sixth	$C1 \to A^\flat 1$	5 : 8
Major Sixth	$C1 \to A1$	3 : 5
Octave	$C1 \to C2$	1 : 2

Through this lens, harmony, and all musical concepts that follow (e.g., consonance, dissonance) are indistinguishable from ratios. The sound of a major triad, which is composed of a major third followed by a minor third (e.g., C, E, G), is an auditory equivalent to the ratio $4 : 5 : 6$. This connection between integer ratios and the frequencies of notes is key to our musical interpretations in Section 3.

A more commonly applied tuning system is *equal temperament*. Just intonation poses several practical difficulties when tuning instruments and performing music. The most jarring quirk is that the frequency ratio between consecutive notes is not constant. Equal temperament avoids these difficulties by preserving the ratio of the frequencies of consecutive notes. In addition to its practicality, equal temperament meaningfully approximates the intervals of just intonation. Now that we have discussed the notes and their placement, let's cover the necessary compositional techniques.

Serialism is a compositional framework largely credited to A. Schoenberg in the early 1900s. The first form of Schoenberg's serialism was *twelve tone music*. In an effort to transcend the limits a tonal system places on music, composers like Schoenberg began exploring music with *no* tonal center (i.e., *atonality*). The twelve tone system provides a formula for writing atonal music. Loosely speaking, to ensure all notes are of equal importance, the system requires that a pitch class, once sounded, cannot be sounded again until the other eleven pitch classes have been sounded. In this system, it is common to refer to pitch classes not by their letter names, but, instead, by integer values between 0 and 11. We highlight this assignment in Table 2.

Table 2: *The Pitch Classes.*

Note Name	C	$C^\sharp/D^\flat$	D	$D^\sharp/E^\flat$	E	F	$F^\sharp/G^\flat$	G	$G^\sharp/A^\flat$	A	$A^\sharp/B^\flat$	B
Integer Value	0	1	2	3	4	5	6	7	8	9	10	11

To begin writing one of these pieces, one only needs to decide on the order of the twelve pitch classes. This sequence is known as a *tone row*, or simply a *row*. To expand on this source material, one is also permitted to use interval preserving transformations of the original row. To hear an example of twelve tone music, we encourage the reader to listen to Schoenberg's *Klavierstück*, Op. 33a (see [8]). For our purposes, serialism establishes a historical precedent, and a meaningful compositional framework, for musical works deriving from core repeating patterns. Next, we derive the musical analogues of the mathematical information that encodes Sierpiński numbers. Recall, Sierpiński numbers are determined by a *covering* and the corresponding *prime covering set*.

Mathematical Concepts and Musical Analogues

The first mathematical concept that we will derive a musical analogue for is the *congruence class* (or *congruence*). Let r and m be integers with $0 \leq r < m$. The congruence $r \pmod{m}$ (read "r mod m") consists of all integers of the form $r + tm$ where t is an integer. We call r the residue and m the modulus of the congruence. As an example, consider the set of positive integers equivalent to 1 (mod 4). This includes the integers 1, 5, 9, 13, and so on. Put simply, we are starting at 1 and counting up by 4s. We use the triple bar equals sign ($\equiv$) to denote membership in a congruence class (e.g., $5 \equiv 1 \pmod{4}$).

Congruence classes are vital in the construction of a Sierpiński number. As such, congruence classes are one of the key mathematical concepts for which we wish to produce a musical analogue. The similarity between a congruence class and a rhythm is quite natural. To detail this connection, consider the musical setting of a drum circle. Let a drum circle consist of finitely many players, each playing a distinct rhythm, all to a common metronome. In this setting, 1 (mod 4) is analogous to a member of the circle playing the 1 of each measure in $\frac{4}{4}$ time. Table 3 illustrates this connection to rhythm using the congruences 1 (mod 4) and 2 (mod 3) as examples.

Table 3: *The Rhythm of a Congruence.*

Congruence Class	Positive Integers Satisfying the Congruence	Beats Played	Repeating Rhythmic Part
1 (mod 4)	1, 5, 9, ...	1, 5, 9, ...	♩ 𝄽 𝄽 𝄽
2 (mod 3)	2, 5, 8, ...	2, 5, 8, ...	𝄽 ♩ 𝄽

We will use this interpretation of congruences as rhythms in deriving our musical analogues and in our final composition.

Of course, drum circles typically consist of many players. A *system of congruence classes* is a collection of one or more congruences, and the musical analogue to a congruence system is a drum circle with 1 or more players. We can now describe one of our key terms. A *covering* is a system of congruence classes C such that every integer z satisfies at least one of the congruence classes in C. A simple example of a covering is $C = \{0 \pmod 2, 1 \pmod 2\}$. Since every integer is either even or odd, every integer will satisfy one of the congruences in C.

The musical analogue to a covering is a drum circle such that, on any given beat, at least one player is striking their drum. The covering associated with the Sierpiński number 78557 is $C = \{0 \pmod 2, 1 \pmod 3, 1 \pmod 4, 3 \pmod 9, 11 \pmod{12}, 15 \pmod{18}, 27 \pmod{36}\}$. A visualization of the covering is presented in Table 4. Notice the least common multiple of the moduli in C is 36, so, to fully visualize the pattern of the covering, we need only exhibit 36 consecutive integers. Since there is a dot in each column, Table 4 justifies that C is a covering.

Table 4: *A Visualization of the Covering C.*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$0 \pmod 2$		•		•		•		•		•		•		•		•		•
$1 \pmod 3$	•			•			•			•			•			•		
$1 \pmod 4$	•				•				•				•				•	
$3 \pmod 9$			•									•						
$11 \pmod{12}$											•							
$15 \pmod{18}$															•			
$27 \pmod{36}$																		

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
$0 \pmod 2$		•		•		•		•		•		•		•		•		•
$1 \pmod 3$	•			•			•			•			•			•		
$1 \pmod 4$			•				•				•				•			
$3 \pmod 9$			•									•						
$11 \pmod{12}$					•												•	
$15 \pmod{18}$															•			
$27 \pmod{36}$									•									

Moving forward, we will refer to this specific covering as C. Rhythmically, C encodes seven parts, one for each congruence. To distinguish the parts, we will assign different pitches to each. By setting the rhythm corresponding to $0 \pmod 2$ to the note $F3$, and assigning each of the subsequent parts to diatonic thirds in the key of C major, we arrive at the piece of music scored in Figure 1 below (listen here [3]).

Figure 1: *The Musical Analogue (up to pitch) of the Covering C.*

The rhythms of the musical lines in Figure 1 capture the information of the covering C, but we chose the pitches arbitrarily. Of course, we wish to minimize the number of arbitrary choices in our musical interpretations — our choices should be as canonic as possible and derive naturally from the source material. To arrive at the pitches, we turn to the *prime covering set*. The prime covering set of $k = 78557$ is $\{3, 5, 7, 13, 19, 37, 73\}$. To remind the reader, this means that the expression $k \cdot 2^n + 1$, for any positive integer n, is divisible by one of these primes. Moving forward, we will refer to this set as $\mathcal{P}$.

Recall musical intervals correspond to whole number ratios. By associating our smallest prime 3 to a note, the rest of our primes also correspond to notes via the ratios 3 : 5 : 7 : 13 : 19 : 37 : 73. Let's assign the prime 3 to the note $C2$ with an approximate frequency of 65.41 Hz. To arrive at a playable piece of music, we will round the resulting frequencies to the closest existing note under equal temperament. We arrive at the correspondence given in Table 5.

Table 5: *The Associated Pitch for each Prime in $\mathcal{P}$.*

Prime	Frequency (Hz)	Approximate Note	Pitch Class
3	65.41	$C2$	0
5	109.02	$A2$	9
7	152.62	$E^{\flat}3$	3
13	283.44	$D^{\flat}4$	1
19	414.26	$A^{\flat}4$	8
37	806.72	$G5$	7
73	1591.64	$G6$	7

Our work yields a chord, which we will refer to as the *Sierpiński chord*, and a partial tone row [0,9,3,1,8,7], which we will refer to as the *Sierpiński row*. In Figure 2, we present the full Sierpiński chord.

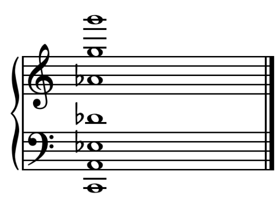

Figure 2: *The Sierpiński Chord.*

One may listen to the Sierpiński chord arpeggiated and then played in full here [4]. Now that we have musical analogues for the covering C and the prime covering set $\mathcal{P}$ (i.e., the information that fully encodes our Sierpiński number), we may produce an auditory analogue to the smallest known Sierpiński number.

We have the rhythms implied in the covering C and the pitches implied by the prime covering set $\mathcal{P}$. There is a bit more information about the construction of a Sierpiński number that dictates how we should assign the pitches to the rhythms. Recall the expression $k \cdot 2^n + 1$ is composite for all positive integers n. More specifically, it will be divisible by one of the primes in the prime covering set $\mathcal{P}$. The prime that divides $k \cdot 2^n + 1$ is determined by the congruence class that n satisfies in the covering C. Explicitly, for $k = 78557$, we have the following implications.

- If n satisfies 0 (mod 2), then 3 divides $k \cdot 2^n + 1$.
- If n satisfies 1 (mod 3), then 7 divides $k \cdot 2^n + 1$.
- If n satisfies 1 (mod 4), then 5 divides $k \cdot 2^n + 1$.
- If n satisfies 3 (mod 9), then 73 divides $k \cdot 2^n + 1$.
- If n satisfies 11 (mod 12), then 13 divides $k \cdot 2^n + 1$.
- If n satisfies 15 (mod 18), then 19 divides $k \cdot 2^n + 1$.
- If n satisfies 27 (mod 36), then 37 divides $k \cdot 2^n + 1$.

We will justify one of the implications above. Since C is a covering, each positive integer n will satisfy a congruence in C. Suppose $n \equiv 1 \pmod 3$ and write $n = 1 + 3t$ for some integer $t \geq 0$. The implication above claims 7 divides $k \cdot 2^n + 1$. Note $2^3 \equiv 1 \pmod 7$, and $k = 78557 \equiv 3 \pmod 7$. A bit of modular arithmetic shows $k \cdot 2^n + 1 \equiv 3 \cdot 2^{1+3t} + 1 \equiv 3 \cdot 2 \cdot (2^3)^t + 1 \equiv 6 + 1 \equiv 0 \pmod 7$ as claimed. This exercise gives insight into the construction of a Sierpiński number, and, for full details, the curious reader may reference [1].

Returning to our musical interpretation, we are not left to our creative whims as far as how to assign our set of pitches to our set of rhythms. The relationship is inherent to the Sierpiński number 78557. We assign the pitch associated with the prime 3 ($C2$) to the rhythm associated with $0 \pmod 2$, as the prime 3 divides the expression $k \cdot 2^n + 1$ while $n \equiv 0 \pmod 2$. Proceeding in this fashion for each congruence and associated prime, we produce the musical analogue to the smallest Sierpiński number. We present the score in Figure 3, and the reader may listen here [5].

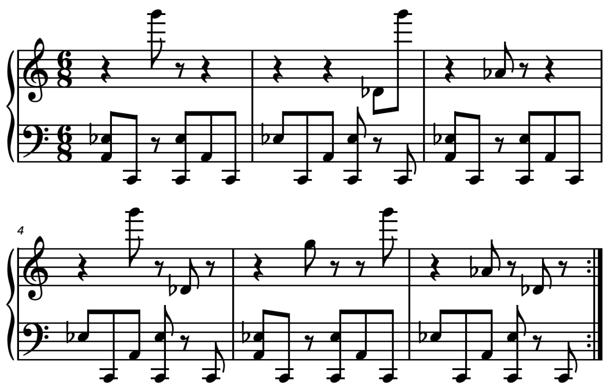

Figure 3: *The Smallest Known Sierpiński Number as Music.*

Before continuing, let's detail our methodology in a general setting to highlight which elements of our result depend directly on the Sierpiński number and which are left to choice. Let k_0 be a Sierpiński number with corresponding covering C_0 and prime covering set $\mathcal{P}_0$. Further, let L_0 denote the least common multiple of the moduli in C_0, and let N_0 denote the number of congruences in C_0. Our methodology interprets C_0 as N_0 rhythmic parts which form a beat pattern of period L_0. The speed at which this pattern is executed (i.e., the tempo) is left to choice and not dictated by our methods. Given a starting pitch, our methodology produces a unique sequence of notes from the prime covering set $\mathcal{P}_0$. Note, our methodology does not dictate the starting pitch, and this is something that must be chosen. Of course, beyond these parameters, there are several elements of performance that are not addressed (e.g., phrasing, articulation, dynamics). We leave these more artful elements up to the performer. In summation, each Sierpiński number that derives from a covering and prime covering set can be associated with a unique piece of music (up to the elements of choice detailed above), just as 78557 is associated with Figure 3.

It is worth nothing that, though our methods can be applied generally to the coverings and prime covering sets of Sierpiński numbers, we simply will not end up with a palatable piece of music in most cases. As we saw in the case of the other Sierpiński number, $k = 5511380746462593381$ is quite large. This k corresponds to a covering with a large least common multiple and a prime covering set which includes large prime numbers. So, when deriving rhythms, we end up with patterns that repeat only after hundreds, thousands, or tens of millions of beats. It is difficult for the human ear to recognize patterns of this length. Further, when deriving pitches from the primes, the primes become so large that the resulting frequencies can not be heard by the human ear, no matter how we choose the starting frequency. The smallest Sierpiński number is one of finitely many cases we can apply this methodology to derive an auditory analogue. It is serendipitous that the rhythm implied by the covering associated with $k = 78557$ repeats every 36 beats (this a perfectly reasonable musical amount of time) and that the ratios implied by the prime covering set all yield notes within the range of human hearing. Now that we have derived the musical analogues to the mathematical concepts inherent to the Sierpiński number 78557, we will use them to compose a piece of music.

The original composition that follows is an artistic endeavor. We employ the compositional techniques of serialism to the analogues derived above (Figure 2 and Figure 3), but our compositional choices within this setting are entirely up to taste. Before presenting the final work, a short piece scored for piano, we will describe our process, and how our compositional choices reflect the core mathematical material. We invite the reader to reference the score (see [7]) as we proceed.

In the bass clef of measure 2, the *Sierpiński row* (scored in Figure 4 below) is introduced as the foundational bass line of the piece.

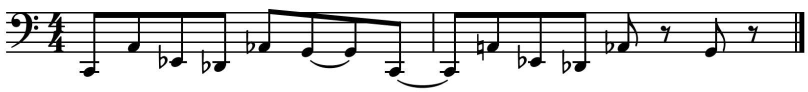

Figure 4: *The Sierpiński Row as a Bass Line.*

In the treble clef, the 36 eighth note pattern, as dictated by the covering, cycles several times. We begin with only a few congruences in the presentation, and we add more with each cycle. The first presentation of the covering begins on the 5th eighth note of measure 6, and it only contains the congruences 0 (mod 2), 1 (mod 3), and 1 (mod 4). It associates the pitch classes C, A, and $E^\flat$ to these rhythms respectively. In presenting the 36 eighth note pattern, we will count from 0 to 35 rather than from 1 to 36. By choosing the 5th eighth note of measure 6 to be beat 0 of the pattern, the congruence $0 \pmod 2$ dictates we sound a C here. Continuing, the 1st eighth note in measure 7 now corresponds to the 4th beat in the pattern. Accordingly, the first C note in the treble clef of measure 7 is corresponding to $4 \equiv 0 \pmod 2$, and the simultaneous A being played corresponds to $4 \equiv 1 \pmod 3$. As a comment, counting the eighth notes between instances of a particular pitch class in the treble clef is a quick way to identify which congruence you are looking at. Notice how the pitch class A appears every 3 eighth notes after its entrance in measure 7 until it ceases.

Through the piece, we build the covering over several cycles (to see where these cycles start, begin by looking for a C followed by A and $E^\flat$ played simultaneously). Accordingly, the right hand is full of syncopation toward the start of the piece, indicating the holes in the covering. By the end of the piece, these holes are filled with yet unheard intervals, accenting them to the ear. The final cycle of the full covering begins at measure 24 and is expressed over a bass part deriving from the retrograde form (i.e., played in reverse) of the Sierpiński row. Measures 24 though 28 are isolated in Figure 5 below.

Figure 5: *The Fully Realized Covering over the Retrograde Sierpiński Row.*

The retrograde Sierpiński row begins in the bass clef on the 2nd eighth note of measure 24, while, in the treble clef, the first note of C again corresponds to $0 \pmod 2$. Notice every eighth note in the treble clef, up to the 36th eighth note after the cycle starts, is being sounded — every integer is being covered! The full covering concludes on the 4th eighth note of measure 28. As expected, the final $D^\flat$ corresponds to $35 \equiv 11 \pmod{12}$. Just after Figure 5 above, the piece concludes with a measure of $0 \pmod 2$ before sounding a final whole note. One may listen to the MIDI performance here [6]. We title our piece after the efforts toward the Sierpiński Problem. We call it *Seventeen or Bust.*

Though producing a composition is the natural conclusion to our work, we consider the analogues used as the source material to be our primary academic contribution. The source material for this composition includes the rhythms implied by the covering C (Figure 3) and the Sierpiński chord (Figure 2). We invite any interested composers to use these materials as they see fit and to share any resulting compositions with us.

Summary and Conclusion

We present a methodology of producing auditory analogues of mathematical objects that derive from coverings which may be extended to other settings involving congruence classes. In applying our methods, we produce pitched rhythmic patterns from the smallest known Sierpiński number, $k = 78557$. We use the resulting auditory analogues as the compositional source material for an original serial composition.

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[6] R. Groth. Original Composition Audio.
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