Gosper World: A Hexagonal Map Using Gosper Fractals

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Abstract

This paper presents a map projection based using Gosper Island fractals to create a constant area world map with hexagonal-like tiles. The tile's self-similarity facilitates rendering the map at various scales and resolutions. The globe is projected into into a rhombic dodecahedron, which is then unfolded and transformed into a map in a hexagonal grid which can be rearranged into multiple configurations.

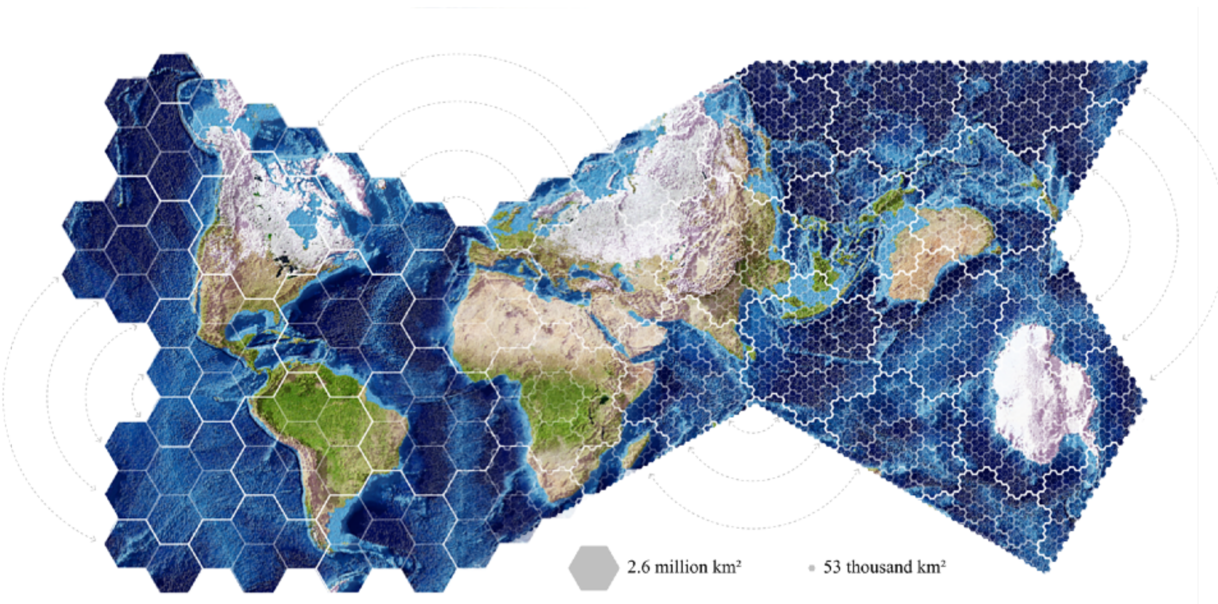
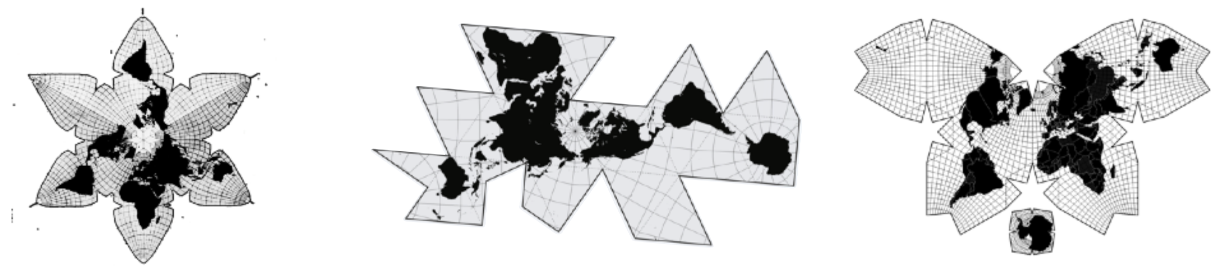


Figure 1: *The Gosper world map.*

Introduction

Most maps are designed to say something about the world, but some are designed to say something about maps. This paper is a case of the latter. While it is true that the mark of a good tool is that it doesn't require you to think about it, when we normalize the distortions that most everyday maps carry, we risk internalizing an incorrect view of our own planet and our place in it. While most maps are often presented in a rectangular or oval form which will necessarily carry some extreme distortions on the edges, some remarkable maps have avoided that by creating a more fragmented overall shape. Notable examples of these would be the Cahill's "Flower", Buckminster Fuller's Dymaxion map and Waterman's Butterfly projections (see Figure 2).

The map presented on Figure 1 is just one configuration of a more general system. Other layouts will be used later in the paper and even more more renderings can be downloaded in the supplemental material.

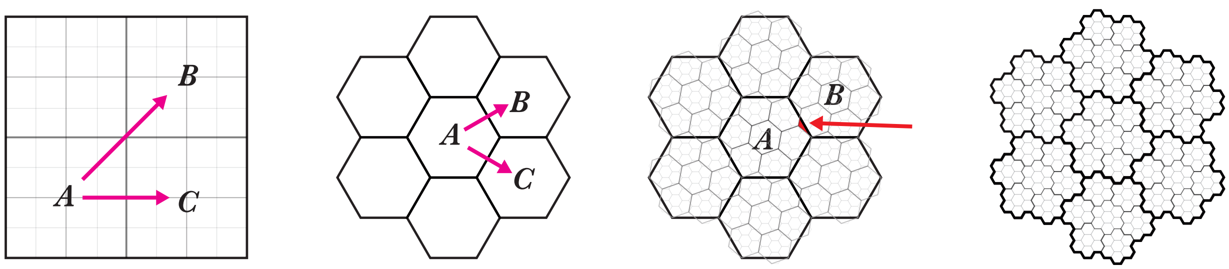


(a) (b) (c)

Figure 2: *(a) Cahill's "Flower", 1909, (b) Fuller's "Dymaxion", 1954 (c) Waterman's "Butterfly", 1996*

Gosper Island Grid

Most coordinate systems use a square grid, for among its characteristics, its ease of subdivision into smaller subgrids. Hexagonal grids are a common alternative, usually chosen for the fact that they offer uniform distances among their neighbors, facilitating approximations of calculations of radius and circular areas (see Figure 3a and 3b). However, a hexagonal grid does not subdivide very easily: while there are many ways to divide a single hexagon into smaller shapes, these will not overlap perfectly with their parents [6, 7], creating points on the map that might belong hierarchically to two different roots, depending on which level you are considering (see the small red area highlighted in Figure 3c). A good compromise between these two grid types is to use Gosper Islands (Figure 3d), a fractal with hexagonal symmetry which offers both a grid with six uniformly distanced neighbors and a perfectly self-similar subdivision of cells [4].



(a) (b) (c) (d)

Figure 3: *(a) square grid, (b) hexagonal grid, (c) hierarchical hexagonal grid, (d) Gosper Island*

A notable example of a hierarchical hexagonal grid is Uber's "H3", which uses such a system to store and analyze its global data on car rides [1]. However, "H3" does not use solely hexagons; it employs a gnomonic projection centered on an icosahedron. The 12 vertices of the icosahedron are transformed into 12 pentagonal faces by carving the 20 triangular faces into a great number of hexagons, depending on the desired map resolution. The positions of the pentagons are based on Fuller's Dymaxion, which avoids major landmasses; however, the existence of the pentagons might not be appropriate for all use cases. For example, a video game designer who wants to display a world map using solely hexagonal tiles would have to keep these pentagons inaccessible, and a map designer who wishes to show the flattened map with only hexagons might have to 'fill in' blank areas with non-existent oceans [5]. The Dymaxion overall shape, while iconic, also might have too many acute corners and empty spaces for some applications.

Rus [3] demonstrated that it's possible for a grid of pure hexagons to fold into polyhedrons, folding four hexagons into both an octahedron and a tetrakis hexahedron. In this process, some hexagons are bent or connect to the same neighbor twice, meaning it's not a polyhedron made of hexagonal faces, but rather a hexagonal network folded around the faces of a solid. This paper applies Rus' method to the rhombic dodecahedron (Figure 4a), a Catalan solid known for its space-tiling properties, analogous to how hexagons tile a plane. Both shapes can be created by putting pyramids on the faces of a cube, so by manipulating the height of the pyramid (therefore the face's length), one can be transform into the other.

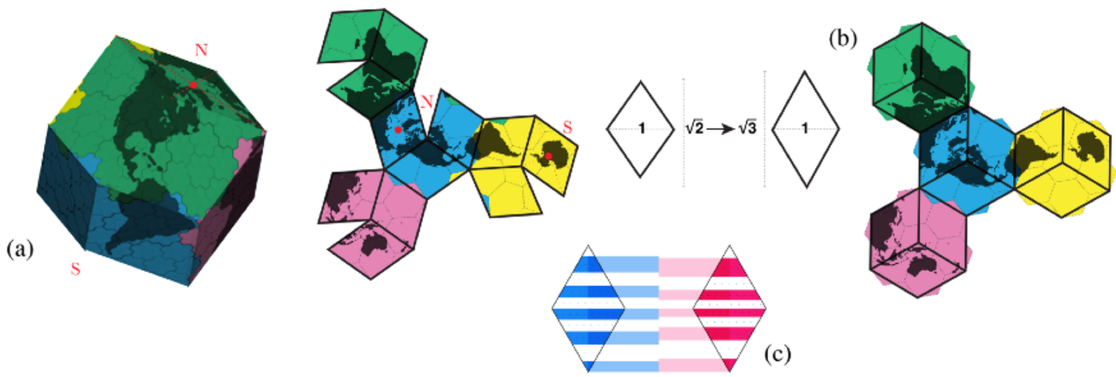


Figure 4: *an unfolded rhombic dodecahedron and its transformation into a hexagonal grid*

The globe is projected using Furuti's rhombic dodecahedron map [2], with the poles located at the centers of the top and bottom faces (Figure 4a). Each rhombic face is then stretched along the longer diagonal by a factor of $\sqrt{2}/\sqrt{3} \approx 0.8164$ so that they form golden rhombi (composed of two equilateral triangles), allowing them to be arranged into four hexagons (Figure 4b). This process creates less distortion than a direct gnomonic projection on a tetrakis hexahedron (see Figure 4c, and refer to the supplemental material for details). The result is a set of four Gosper islands (or simply hexagons if preferred) that can be subdivided and rearranged to create a reconfigurable map, in the spirit of Lee's Tetrahedral or the award-winning Authagraph projection [3].

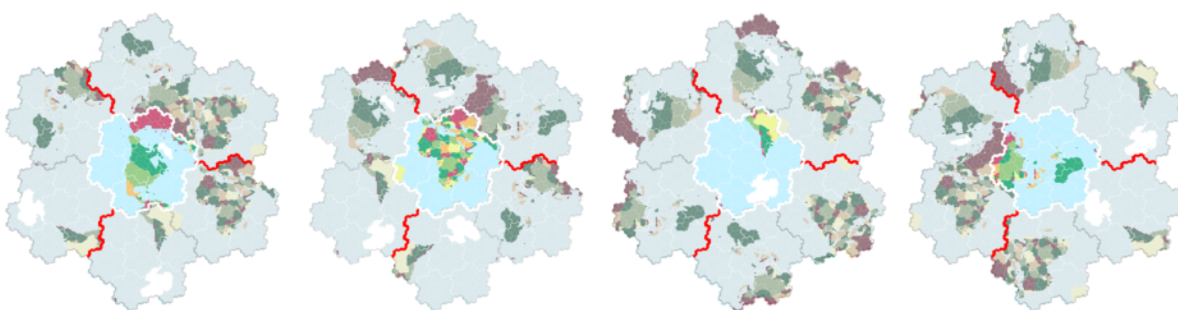


Figure 5: *The four main Gosper tiles and the position of their neighbors. Red lines indicate a discontinuity. The overall shape is also a Gosper Island.*

The hexagonal nature of the tiles allows for easy comparisons of area sizes (Figure 1 demonstrates the areas of the hexagons) as well as calculating simplified routes that traverse the globe, including transpolar routes, as seen in Figure 5. Since the main tiles are hexagonal and not triangular, the result tends to have fewer empty spaces compared to those based on an icosahedral projection.

A common way to demonstrate map distortion is through a Tissot indicatrix, where a series of equally sized circles are overlaid on the map. As shown in Figure 7 (on the next page), the Gosper World map renders all circles as squished ovals, but of similar size. In other words, overall distortion is not concentrated on the edges; instead, area distortion is minimized while angular distortion is kept mostly uniform.

Other applications

The method can also be applied to panoramic images as a way to present them as tileable hexagons (Figure 6). Further applications for a globe using fractal tiles could include simulations such as cellular automata that can be calculated at higher precisions in areas of interest but then applied at a coarser resolution on the rest of the globe so that small local phenomena can affect the whole planet and vice versa.

Figure 6: *panorama of Celestine Palace, Lecce, Italy*

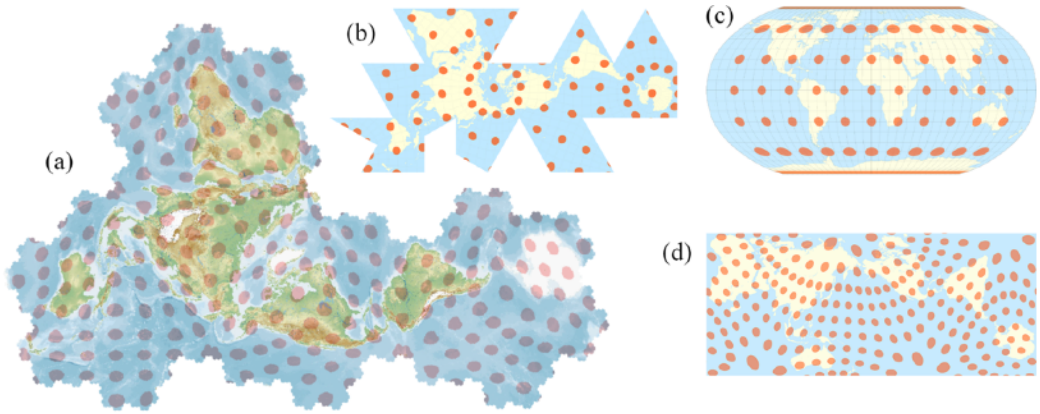


Figure 7: *Tissot indicatrix: (a) Gosper World (b) Dymaxion (c) Robinson Projection (d) Autagraph*

Conclusion

There are many applications for an area-accurate world map using only hexagons: an infographic that displays some value per area using dots; a video game that uses only hexagonal art assets yet wants to have a spherical world; a board game that allows easy movement of units across an accurate world map; a classroom puzzle that helps students create their own map that matches their perspective. In these use cases, a designer will often overlay a hexagonal grid on a standard rectangular map (retaining the underlying distortion), or in some cases base their work on a Dymaxion map but fill the empty spaces with ocean. It is my hope that this work presents an interesting alternative for map designers everywhere, as well as a fun new tool for teachers of all kinds

Acknowledgments

The underlying data for the maps shown are based on equirectangular maps, all using open licenses. NASA's Blue Marble is licensed as Public Domain, and the following authors have licensed their images under the Creative Commons Attribution-Share Alike 4.0: Gundan, Tobias Jung, Supremeaesthete, cptNautilus, Eric Gaba, and DrTrumpet. The AuthaGraph map is copyrighted by AuthaGraph Co., but the Tissot presented here is a MIT-licensed recreation by jkunimune. I'd also like to thank Felipe Van de Sande Araujo from NTNU, without whose feedback this paper in its current form would not exist.

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