Constructing the Snub Cube via Intersection of Generalized Steinmetz Curves

Luís Mateus

CIAUD, Centro de Investigação em Arquitetura, Urbanismo e Design, Faculdade de Arquitetura, Universidade de Lisboa, Portugal; lmmateus@fa.ulisboa.pt

Abstract

The Snub Cube can be obtained using various methods like paper folding, snubification, physical equilibrium simulation or through the Tribonnaci constant. These methods imply the application of formulas to get the coordinates of the polyhedron. We present a novel method that can be applied directly in any 3D modelling tool, expanding the notion of geometrical construction, without the need for preset formulas or constraints. This can be particularly useful for non-experts like artists, architects, or designers. The method involves deriving the vertices of the polyhedron by intersecting generalized Steinmetz curves. However, to demonstrate the method, we made an analytical development. A similar approach can be applied to the Snub Dodecahedron.

Introduction

The Snub Cube is one of the thirteen Archimedean polyhedra. It cannot be constructed geometrically if, by this, we consider as geometrical constructions only those that can be performed with ruler and compass. This polyhedron can be obtained via paper folding [2], snubification [3], through physical equilibrium simulation [1] and using the Tribonacci constant [5]. In our view, we can extend the idea of geometric construction to any geometric configuration that can be obtained via interaction of any geometrical objects. This is even more pertinent in a world of computers and 3D modelling software. In this paper we present a new method to construct the Snub Cube using the intersection of cylindrical surfaces with axes meeting in the centre of the polyhedron, which can be applied directly with any modelling tool by any non-expert like an artist, an architect, or a designer. Then, the approach is developed analytically to demonstrate it.

Generalized Steinmetz Curves

A Steinmetz curve s is the result of the intersection of two cylindrical surfaces of revolution with axes mutually perpendicular [4]. For this purpose, let us consider two surfaces, one with vertical axis z, and radius p, and the other with horizontal axis x, and radius q (Figure 1(a)). This curve can be generalized to the case where the axis of the cylinder of radius q is oblique to axis x by an angle α (Figure 1(b)), and to the case where this axis further rotates around the axis z by an angle β (Figure 1(c)).

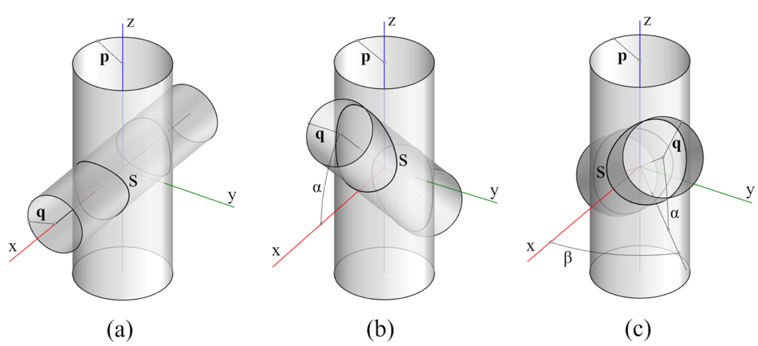


Figure 1: *The intersection of cylinders with intersecting axes: (a) Steinmetz curve, (b)generalization when the axis of the cylinder of radius q is oblique to x, (c) generalization when the axis of the cylinder of radius q is further rotated around z.*

The parametric equations of the Steinmetz curve (Figure 1(a)) are the following:

(1) $X = p\cos(t)$

(2) $Y = p\sin(t)$

(3) $Z = \pm\sqrt{q^2 - p^2 sin^2(t)}$

The parametric equations of the generalized curve, when the axis is oblique to x (Figure 1(b)), are the same for X and Y, and the following for Z:

(4) $Z = \frac{p\cos(t)\sin(\alpha) \pm \sqrt{q^2 - p^2 sin^2(t)}}{\cos(\alpha)}$

If the axis is further rotated around z (Figure 1(c)), the parametric equations are again the same for X and Y, and the following for Z:

(5) $Z = \frac{p\cos(t-\beta)\sin(\alpha) \pm \sqrt{q^2 - p^2 sin^2(t-\beta)}}{\cos(\alpha)}$

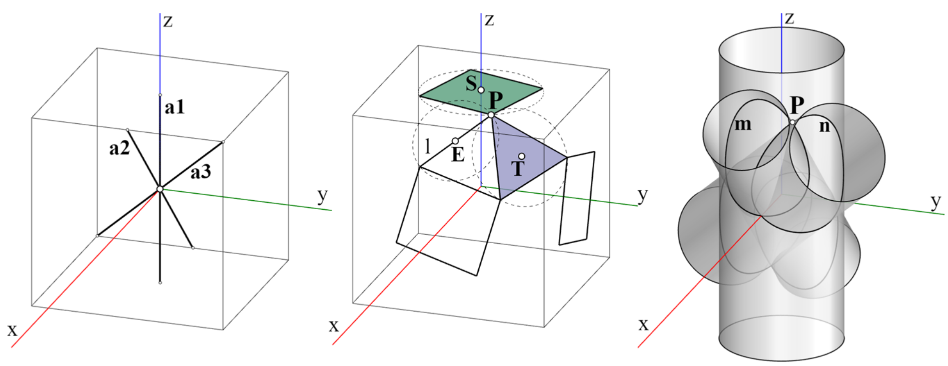
Relating the Snub Cube and the Generalized Steinmetz Curves

Considering the circumscribed cube, we can define the types of axes of chiral symmetry of the Snub Cube. In Figure 2(a), one exemplar of each of these types is defined. The axes of type $a1$ intersect, perpendicularly, the square faces at their centers in points like S. The axes of type $a2$ intersect, perpendicularly, the edges of type l at their midpoints in points like E. The axes of type $a3$ intersect, perpendicularly, the triangular faces, that have vertices in three different squares, at their centers in points like T (Figure 2(b)). For simplicity, considering $L = 2$ as the length of the edges of the Snub Cube, these are, respectively, axes of cylindrical surfaces, that pass through vertices of the polyhedron, with radii:

(6) $e = \frac{L\sqrt{2}}{2} = \sqrt{2}$

(7) $f = \frac{L}{2} = 1$

(8) $g = \frac{L\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$



(a) (b) (c)

Figure 2: *Snub cube and Steinmetz curves: (a) exemplars of the axes of chiral symmetry of the Snub Cube ,(b) points of intersection of the axes with the square face (S), triangular face (T) and edge (E), (c) vertex P of the Snub Cube laying on the intersection of the generalized Steinmetz curves m and n.*

The cylindrical surfaces with axes $a2$ and $a3$ intersect the cylindrical surface with axis $a1$ according to the curves m and n, respectively. The vertex P of the Snub Cube is one of the intersection points of the curves m and n (Figure 2(c)). Notice that the intersection curves m and n have two branches but only one is considered. In the intersection of the other branches lies a point analogous to P that belongs to the mirrored version of the Snub Cube. The other vertices of the polyhedron, and its mirrored version, can be generated by properly changing the orientation of the cylinders. This approach can also be applied to the Snub Dodecahedron.

For curve m, where $\alpha = \pi/4$, the parametric equations can be obtained from equations (1), (2) and (4), making $p = e$ and $q = f$, which, after developing, results in:

(9) $$X = \sqrt{2}\cos(t)$$

(10) $$Y = \sqrt{2}\sin(t)$$

(11) $$Z = \sqrt{2}\cos(t) \pm 2\sqrt{\frac{1}{2} - sin^2(t)}$$

Similarly for curve n, where $\beta = \pi/4$, $cos\beta = sin\beta = \sqrt{2}/2$, $\alpha \approx 0.61547970867039...$, $cos\alpha = \sqrt{6}/3$ and $sin\alpha = \sqrt{3}/3$, the equations are the same as (9) and (10) for X and Y. The equation for Z can be obtained from (5), making $p = e$ and $q = g$, which, after developing, results in:

(12) $$Z = \cos\left(t - \frac{\pi}{4}\right) \pm \sqrt{2 - 3sin^2\left(t - \frac{\pi}{4}\right)}$$

Determining the Coordinates of Vertex P

Determining the coordinates of vertex P is enough because the remaining coordinates of the polyhedron can be obtained by permutating properly their values and signs. Regarding curve m, equation (11), rewritten only in terms of $cos(t)$ becomes:

(13) $$Z = \sqrt{2}\cos(t) \pm \sqrt{4cos^2(t) - 2}$$

Similarly, regarding curve n, equation (12), rewritten and simplified only in terms of $cos(t)$ becomes:

(14) $$Z = \frac{\sqrt{2}}{2}\cos(t) + \frac{\sqrt{2 - 2cos^2(t)}}{2} \pm \sqrt{\frac{1}{2} + 3\cos(t)\sqrt{1 - cos^2(t)}}$$

Now, we can replace Z, in equation (13), by the right side of equation (14), and because the point P is in the upper parts of curves m and n, we will consider the positive roots in the right sides of equations (13) and (14). If the negative roots are considered, then the corresponding point P would be in the opposite face of the circumscribed cube and would belong to the mirrored Snub Cube. Moreover, we will consider $x = cos(t)$ in the following equation:

(15) $$\sqrt{2}x + \sqrt{4x^2 - 2} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2 - 2x^2}}{2} + \sqrt{\frac{1}{2} + 3x\sqrt{1 - x^2}}$$

Developing equation (15) will result in:

(16) $$x^6 - x^4 + \frac{1}{2}x^2 - \frac{1}{4} = 0$$

Letting $x^2 = u$, in equation (16), yields:

(17) $$u^3 - u^2 + \frac{1}{2}\mathrm{u} - \frac{1}{4} = 0$$

This is a cubic equation in the form of $au^3 + bu^2 + cu + d = 0$ that can be solved applying the Cardano formula. Making $a = 1$, $b = -1$, $c = 1/2$ and $d = -1/4$, the real root of equation (17) can be determined:

(18) $$u = \frac{1}{3} + \sqrt[3]{\frac{17}{216} + \sqrt{\left(\frac{17}{16}\right)^2 + \left(\frac{1}{18}\right)^3}} + \sqrt[3]{\frac{17}{216} - \sqrt{\left(\frac{17}{16}\right)^2 + \left(\frac{1}{18}\right)^3}}$$

Since $u = cos^2(t)$, then, making $v = 1 - cos^2(t) = sin^2(t)$, results in:

(19) $$v = \frac{2}{3} - \sqrt[3]{\frac{17}{216} + \sqrt{\left(\frac{17}{16}\right)^2 + \left(\frac{1}{18}\right)^3}} - \sqrt[3]{\frac{17}{216} - \sqrt{\left(\frac{17}{16}\right)^2 + \left(\frac{1}{18}\right)^3}}$$

From where $\cos(t) = \pm\sqrt{u}$ and $\sin(t) = \pm\sqrt{v}$. And, if we consider only the positive roots, then:

(20) $$t \approx 0.49798501463967 \ldots$$

Plugging in t in equations (9), (10) and (11) or (12), will return the coordinates of a vertex P of a Snub Cube of edge length $L = 2$. Alternatively, we can plug in directly $cos(t)$ and $sin(t)$ in those equations, which, in the case of equation (12), requires its expansion.

For $L = 1$, the coordinates of the vertex P are:

(21) $$X_P \approx 0.62122641055659 \ldots$$

(22) $$Y_P \approx 0.33775397381375 \ldots$$

(23) $$Z_P \approx 1.14261350892596 \ldots$$

Making $a = X_P$, $b = Y_P$ and $c = Z_P$, the complete set with the coordinates of the 24 vertices of the Snub Cube correspond to the following permutations and sign adjustments of a, b and c, as follows:

(24) $$(a,b,c), (-b,a,c), (-a,-b,c), (b,-a,c), (a,-b,-c), (-b,-a,-c), (-a,b,-c), (b,a,-c)$$
$$(c,b,-a), (c,a,b), (c,-b,a), (c,-a,-b), (-c,-b,-a), (-c,-a,b), (-c,b,a), (-c,a,-b)$$
$$(-b,c,-a), (-a,c,b), (b,c,a), (a,c,-b), (b,-c,-a), (a,-c,b), (-b,-c,a), (-a,-c,-b)$$

These permutations were obtained, first, by rotating the vertex P around the z axis by $\pi/2$, π, and $2\pi/3$. This returns the four vertices in the upper face of the circumscribed cube, which were then rotated by π around the x axis to get the four vertices of the lower face of the circumscribed cube (first line in (24)). After, those eight vertices were rotated by $\pi/2$ around the y axis (second line in (24)), followed by another rotation of $\pi/2$ around the z axis (third line in (24)).

Summary and Conclusions

In this paper we developed a novel method to construct the Snub Cube via intersection of generalized Steinmetz curves. We determined the coordinates of a single vertex and obtained the remaining ones with rotations. This approach is especially suited for non-experts like artists, architects, or designers because it can be implemented just via 3D modelling and without the explicit use of formulas or given coordinates.

Acknowledgements

This work is financed by national funds through FCT - Fundação para a Ciência e a Tecnologia, I.P., under the Strategic Project with the references UIDB/04008/2020 and UIDP/04008/2020.

References

[1] L. Baglioni, F. Fallavolita and R. Foschi. "Synthetic Methods for Constructing Polyhedra." *Polyhedra and Beyond*. Birkhäuser. 2022. https://doi.org/10.1007/978-3-030-99116-6_1.

[2] U. Hartl and K. Kwickert. "Constructing the Cubus simus and the Dodecaedron simum via paper folding." *Geom Dedicata*. vol. 166, Springer. 2013, pp. 1-14.

[3] E. Weisstein. "Snub Cube." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/SnubCube.html.

[4] E. Weisstein. "Steinmetz Curve." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/SteinmetzCurve.html.

[5] "Snub cube." Wikipedia, The Free Encyclopedia, Wikimedia Foundation, 10 January 2024. https://en.wikipedia.org/wiki/Snub_cube.