The Art of Knot Data

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Abstract

Ernst and Sumners' theorem, affirming that knots constitute a form of big data, coupled with the comprehensive knot tabulation by Burton, Hoste, Thistlethwaite, and Weeks, along with numerous computations of knot invariants, establishes the groundwork for employing big data methodologies in knot theory. Utilizing dimension reduction and machine learning methods, such as Ball Mapper, not only yields valuable insights into the statistical characteristics of knots but also offers compelling means to visually represent the intricate space of knots. The appeal of generative art obtained is multifaceted, encompassing both aesthetic appeal and the complexity of mathematical statements.

Knots as Big Data

Knots and links possess a profound presence in the realm of art, spanning across diverse cultures and historical periods including Babylonian, Egyptian, Greek, Chinese, Byzantine, and Celtic traditions. This rich heritage continues to influence modern art, as seen in the sculptures of artists such as A. Brakke, B. Collins, Bathsheba Grossman, C.O. Perry, and R. Roelofs. Some of the recent art relies on advanced computational and technological advances such as 3D printing, while our artwork harvests the power of big data analysis techniques as well as the recent developments in knot theory.

According to the theorem by Ernst and Sumners, the number of distinct knots increases exponentially with the minimal crossing number. This theorem implies that knots are "big data" which is evident from knot tables containing over 350 million knots with up to 19 and almost 1.9 billion with 20 crossings obtained by Burton [1] and Thistlethwaite, respectively. Viewing knots as big data opens new ways of analyzing and visualizing collections of knots through the use of advanced big data visualization techniques, such as Ball Mapper [3], in conjunction with other dimension reduction and machine learning methods [2, 5].

Knot theory offers a way of extracting simpler descriptors of these remarkably complex geometric objects which can be used as input for "big data" techniques. One solution comes in the form of knot invariants, essential tools for distinguishing knots. Knot invariants can be numerical, such as crossing or linking numbers or signature, polynomials such as Alexander, Jones, Kauffman etc. or more algebraically sophisticated such as link homology theories. What all invariants have in common is the aim to capture distinct characteristics that are preserved under isotopic transformations of a knot, thereby providing a mathematical fingerprint of sorts. However, most of the computable knot invariants fail to distinguish all knots. For example, the infinite family of pretzel knots with the Conway symbol $(2k + 1), 3, -3$ have the Alexander polynomial equal to $2 - 5x + 2x^2$ but they are distinguished by their Jones polynomials [4].

Instead of comparing knot invariants via classification tasks we delve into visual exploration of the data obtained from knot invariants such as the Alexander polynomial of all knots with up to 17 crossings, its roots, and several numerical knot invariants such as signature, determinant, and crossing number. Using tools from topological data analysis we address open questions in knot theory showcasing how data-driven approaches can be leveraged to obtain new theoretical insights, (re)discover theorems, and generate art by visualizing relations and robust statistical results. Integration of diverse approaches in solving sophisticated mathematical problems emphasizes a harmonious blend of creativity and structure in mathematics and art [6].

Ball Mapper: Exploratory and Visualization Tool

The inherent complexity often drives the pursuit of simplification to enhance our understanding. Visualization techniques offer a spectrum of methodologies for representing data, with the most prevalent approaches involving linear or nonlinear embedding that aim to minimize a specific function. Notably, nonlinear dimension reduction techniques such as t-SNE or UMAP have gained prominence for their efficacy in preserving the local neighborhood structure of data points. However, a significant limitation of these methods is their tendency to distort the global structure of the data, which often remains elusive. This challenge underscores the value of topological data analysis (TDA) tools, especially those based on mapper-type algorithms, in providing a more comprehensive understanding of data structure.

For a given finite point sample X and $\epsilon > 0$, the Ball Mapper algorithm [3] constructs an undirected graph G, called the Ball Mapper graph of X at the radius ϵ, whose shape captures the essential feature of the shape of X. This is achieved by covering X with a collection of overlapping balls of radius ϵ and assigning a vertex of G to each balls, while the edges represent their non-empty intersections. The Ball Mapper graph G serves as a 1-dim model for the set X. If the point cloud X comes with a function $f : X \to \mathbb{R}$, the Ball Mapper construction can be extended to visualize such function by coloring each vertex of the Ball Mapper by the average value of the function within its corresponding ball of the cover. This approach provides a tool to understand the shapes of complex, high-dimensional data, as well as to analyze functions defined on them.

(a) **(b)** **(c)**

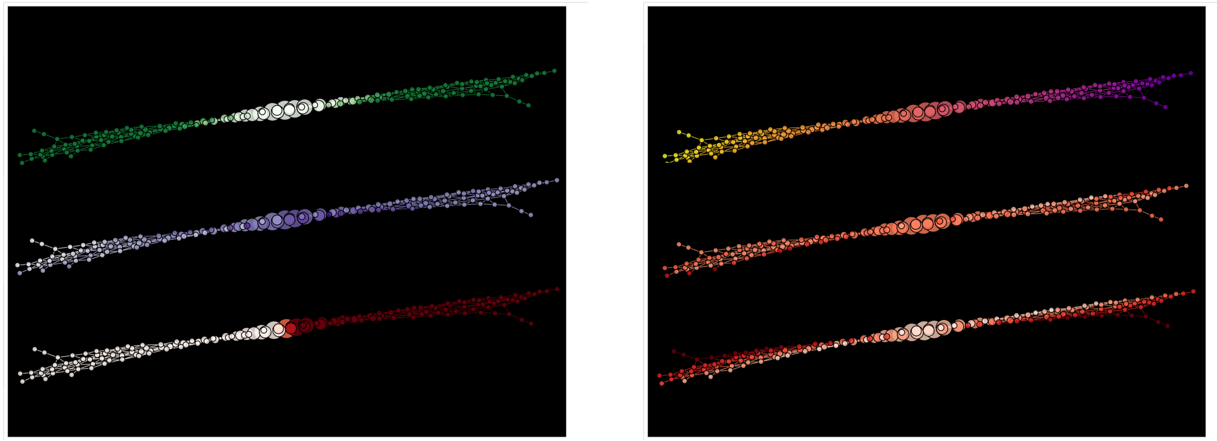
Figure 1: *Alexander data visualized using (a) 2D PCA (top) and Ball Mapper (bottom) (b) 2D UMAP (c) 2D t-SNE projections. All plots colored by signature modulo 4 (white is 0, red is 2).*

Comparison between Ball Mapper, PCA, UMAP, t-SNE on the point cloud obtained from the coefficients of the Alexander polynomial up to 17 crossing knots colored by the signature modulo 4 is shown in Figure 1. Ball Mapper indicates linear structure confirmed by the existence of dominant first principal component Figure 1a, completely different from the ones in Figure 1b, c. The difference is likely due to properties of nonlinear dimension reductions designed to preserve local structure of data sampled from manifolds. Hence, in the case of highly non-generic data such as ours, the outputs do not capture the underlying global structure.

Ball Mapper on Alexander Coefficient Data

Ball Mapper can be used to visualize relations, such as the one between the Alexander polynomial and other knot invariants. Figure 2 shows the Ball Mapper graph of the Alexander data, or Alexander BM for brevity, colored by knot invariants such as the leading coefficient related to the fibredness or the maximal degree that gives knot genus, providing insight into their global behavior. For example, non-alternating knots are concentrated in the middle, the signature and determinant increase from one to the other end, while maximal degree of the Alexander polynomial filters the linear structure.

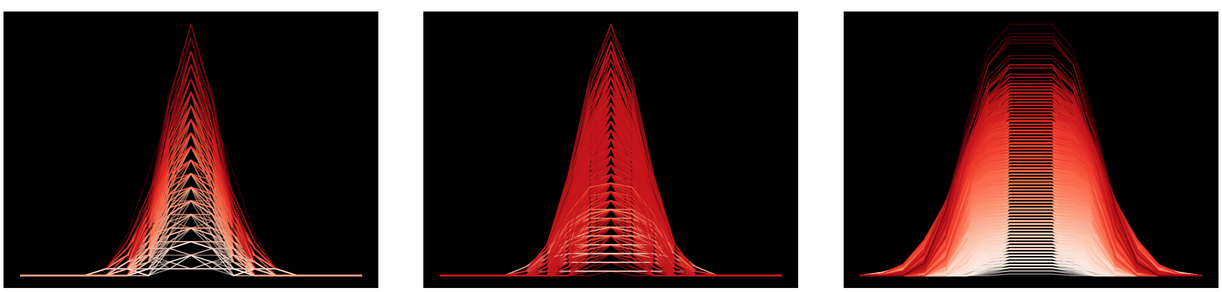
Ball Mapper is not just the visualization tool: it can generate art and hypotheses! The BM graph in the bottom of Figure 2a implies that the Alexander polynomial detects signature mod 4, as one half is white



(a) (b)

Figure 2: *Alexander BM colored by whether the knot is (a) alternating (green) or not (white), signature (purple high, white low), signature mod 4 (red is 2, white is 0); (b) determinant (yellow low, purple high), leading coefficient (white low, red high), maximal degree (white low, red high).*

(zero) and the other red (two). SVM (Support vector machine) found a separating hyperplane normal vector $[1, -1, 1, -1, \ldots]$ in 17-dimensional Alexander data that recovers the theorem stating that the sign of the Alexander polynomial evaluated at -1 determines signature mod 4. The Fox trapezoidal conjecture, open since 1962, states that for an alternating knot the absolute value of coefficients first increases, then there are m coefficients with constant value then decrease in the same way (since it is palindromic). The conjecture is illustrated with elegant images in Figure 3 that relate plots of the Alexander coefficients for various classes of knots and the determinant of the minimal real part of any of the zeros shown in Figure 4a. Figure 3b and 4b middle illustrate the distribution of minimal real parts of any zero and Figure 4c plots polynomials for non-alternating knots all contained in the center of Alexander BM Figure 4b bottom.

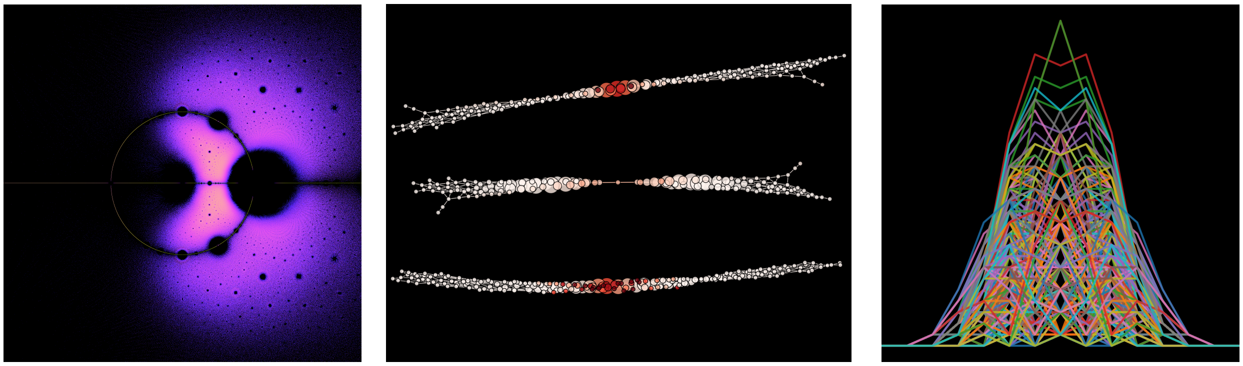


(a) (b) (c)

Figure 3: *(a) Absolute values of the Alexander coefficients of 10 crossing knots and (c) all 3324 alternating knots with $m = 2$ colored by the absolute value of the determinant; (b) the first 500 alternating knots colored by minimum value of the real part of any zero where red is high and white low.*

Summary and Conclusions

In our visually inundated world big data analysis tools provide a way to explore complex, high dimensional, hard-to-sample data, such as the global structure of millions of knots. Applying one such tool to knots, TDA's Ball Mapper [8], comes with a cost: a collection of numerical descriptors derived from knot invariants used



(a) (b) (c)

Figure 4: *(a) Zeros of the Alexander polynomials; (b) Alexander BM for all (top) and for alternating knots (middle) colored by the minimum value of the real part of any zero (using the same color scale, white is 0, red is negative). BM for non-alternating knots (bottom) where white means trapezoidal and red means not; (c) Alexander polynomials of non-alternating non-trapezoidal knots.*

in Ball Mapper does not capture all of knot's properties. In return, we get beautiful images and interactive 3d plots [9], knot landscapes, that capture the statistical, global structure of knot invariants. Utility of Ball Mapper extends beyond knot invariants providing a tool [8] for the math and art community to visualize essential features of high-dimensional data sets, functions on and relations between them.

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