Group Theory-based Dynamic Tiling for Geodesic Dome Design

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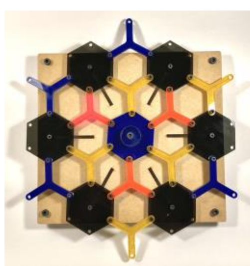
Abstract

This manuscript explores the connection between transformations on geodesic domes and the theory behind spherical patterns-group theory. We demonstrate that abstract concepts in group theory can benefit designers during the early phases of product planning that involve tiling properties. By establishing this link, we hope to provide designers with new tools and insights to create innovative and efficient products.

Introduction

Industrial design is an ever-evolving field that often involves using novel and advanced materials and cutting- edge technologies. In this context, mathematical discussions typically revolve around various measures, such as distances and areas, all connected to geometry. In the present manuscript, we establish a connection between transformations on geodesic domes and the theory behind spherical patterns, specifically group theory. Designers can utilize abstract concepts in group theory to plan products that incorporate tiling properties.


(a)

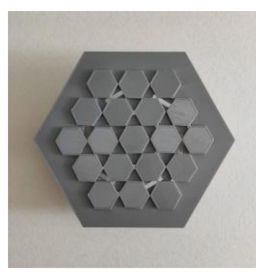

(b)

Figure 1: *Our previous models, define with a proper mechanism a dynamical movement between tilings.*

At the Bridges 2022 art gallery, an isokinetic sculpture based on Hoberman's sphere was displayed, see [4]. The Hex-Flex, see Figure 1(a), sculpture is a mechanical work of art that uses hexagons connected by mechanical movements. All the pieces in the sculpture are derived from a circle with a radius of 'r' inscribed in a hexagon. The connecting pieces that join the corresponding hexagons in motion are obtained by connecting every other vertex. The sculpture's base is designed based on the dimensions of the hexagons and the connecting pieces, demonstrating relative interconnectedness. In our previous paper, "Dynamic Tiling in Industrial Design," see [1], explored the relationships between planar patterns, see Figure 1(b). It demonstrates how dynamic movement can be achieved between two groups of tiling obtained by a regular hexagonal, where reflections, rotation, and translation define the tiling, and its sub-group is determined by rotation and translations. Developed independently, both of these sculptures aim to define dynamic tiling on a plane using distinct hand-operated mechanisms. By leveraging group theory, they showcase transitional mechanical movement powered by rotational input and utilize two different sets of linkage systems.

During the development process, we utilized a range of mathematical and design techniques. For this paper, we selected a spherical geometry that closely resembles Euclidean geometry. In addition, we delved into the world of geodesic domes, which employ spherical designs and repetitive patterns to generate dynamic movement. Following a thorough analysis of blueprints for constructing geodesic domes with repetitive patterns, see [6, 7, 8], we carried out experiments using various materials and mechanisms that could be employed in constructing geodesic domes using different methods. Our approach employs various materials, including cardboard, wooden sticks, wax thread, paper, medium-density fiberboard (MDF), and brass fasteners, each with a distinct purpose contributing to a unique goal. We will manually demonstrate the dynamic motion of the domes, which will help us establish the desired mechanism that transforms between dome types using group rules.

Three-dimensional bodies with a dynamical movement already have been discussed in [3, 5], where a group of mechanisms has been defined to obtain a movement between polyhedral structures. By generalizing the work and changing the Euclidean geometry to spherical geometry, we intend to enhance the design of geodesic domes with creativity and innovation. We aim to imbue them with dynamic properties such as reduction/expansion or exposing/concealing, depending on the requirements of the designer. We adhere to iterative design principles and utilize them as a foundation for construction.

Background on Spherical Patterns

Spherical patterns are obtained by a combination of rotation, and reflection, which leads to the desired translation to tile the sphere. While in the plane, there exist 17 different repetitive tilings (the wallpaper groups), the number of spherical patterns decreases to 14 since we tile a finite surface (compared to plane tilings). There are a few different ways to denote these groups, such as Crystallographic and Coxeter. We chose Conway notations [2], which in our opinion, is the most intuitive notation. The signatures of each of these 14 tiling are denoted by the following notations:

- If there is a reflection respective to the geodesic line, it is denoted by $*$.
- If there is a point of reflection crossing n geodesic lines, it is denoted by $*n$.
- If there is a point of rotation in an angle $\frac{2\pi}{k}$, it is denoted by k.

For symmetry reasons, we can choose a representative for each of the symmetries. Generally, if we write $k * n$, it indicates that the pattern has k rotations and n reflections; or $*mn$ indicates that there are points with m reflections and n reflection respective to another point (for more details see [2]). So, each pattern can be described as a signature that is defined by its respective symmetries. Table 1 lists the signatures obtained in spherical patterns.

Table 1: *All spherical signatures. Every column defines a spherical group relation, where in the first row is the full group, in the last row the smallest subgroup which tiles the sphere.*

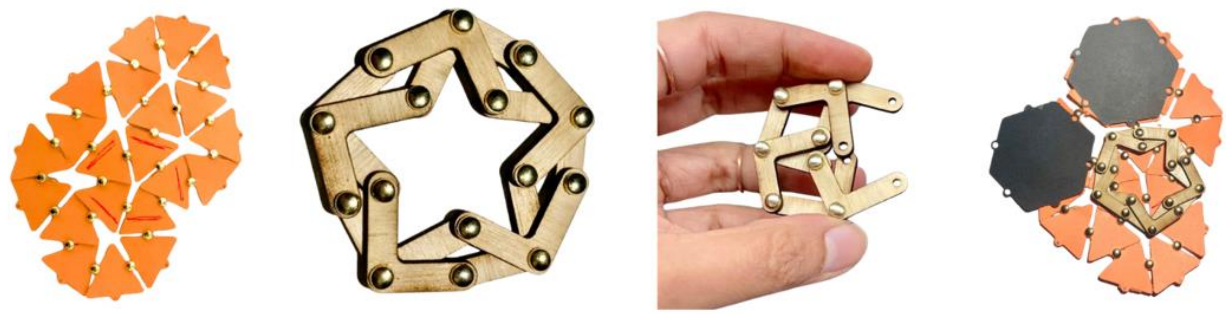
$*532$	$*432$	$*332$	$*22N$	$*NN$
↓	↓	↓	↓	↓
		$3*2$	$2*N$	$N*$
↓	↓	↓	↓	↓
532	432	332	$22N$	NN

Each signature corresponds to a specific symmetry group, with interrelations between the columns. of these signatures has a respective symmetry group, along with relations between each of the columns. For instance, 532 (representing the structure of a soccer ball composed of regular hexagons and pentagons) is a subgroup of $*532$, and 332 is a subgroup of $3*2$, itself a subgroup of $*332$. We can denote these relationships as $532 \leq *532$ and $332 \leq 3*2 \leq *332$, respectively.

Experiments and Results

Before moving to spherical patterns, we first delve into planar patterns, since a planar pattern can be considered to be a local approximation of a spherical pattern. We need to decide which planar pattern and relation is the best deform, according to mathematics and material design.

In Figure 2, dynamic planar patterns implemented with various materials are considered. In Figure 2(a), we use paper and brass fasteners, which are flexible enough materials to obtain a deformation. This approach will give us another avenue in spherical pattern approximation to explore. In Figure 2(b) and Figure 2(c), which emphasize the movement, we use laser-cut MDF and brass fasteners to experiment with a scissor-linkage mechanism for a pentagon that opens and closes. Since in some spherical patterns, pentagons play a major role, we believe that this kind of experiment can lead to the desired mechanism. In Figure 2(d), we combine the previous steps and attempt to approximate from a plane signature a respective spherical one.



(a) (b) (c) (d)

Figure 2: *Exploring movement, deformation, and mechanism.*

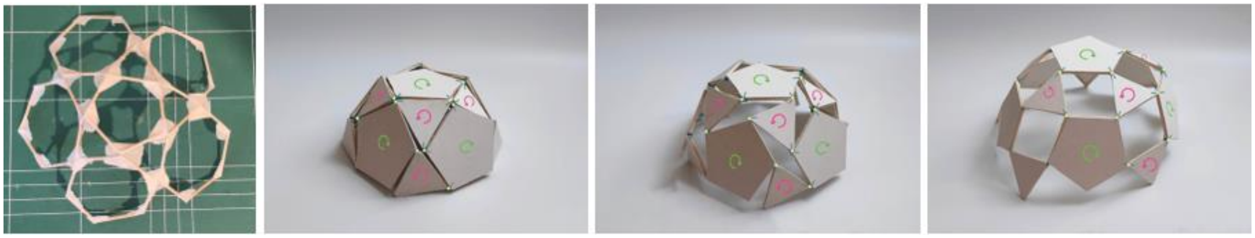
The described exploration above, especially Figure 2(a), which can be deformed easily, led us to think that our initial model needs to be defined by a spherical pattern that is approximated by a collection of hexagons and equilateral triangles (each regular polygon defines a dihedral group with a rotation subgroup) which can be generalized to a pure rotation signature of the pattern. In Table 1, there are five signatures which are defined only by reflection. The plane pattern in Figure 2(a) is with a respective signature of $*632$ which by the deformation to spherical pattern leads to signature $*532$ (the deformations of the plane lead to gaps which leads to the new face of the pentagon in the spherical geometry).

To construct Figure 3(a), which is a local representation of a dome $*532$ made with wooden sticks and masking tape, and by manually turning one of the vertices (sliding) by hand, it showed us that such a dynamic movement from $*532$ to 532 (which is a subgroup) can be obtained on sphere or hemisphere which can be considered as geodesic dome.

The dome in Figure 3(b), with signature $*532$ represents an icosidodecahedron, while the dome in Figure 3(c), with signature 532, represents a snub dodecahedron.

We constructed the dome using cardboard, wooden sticks, wax thread, and repress as shown in Figure 3, marked by signature *532. Figure 3(b) is the initial stage of the dome with a respective signature $*532$. In Figure 3(c), the middle stage is obtained by rotating along the vertices, with a respective signature of 532 and increasing the volume of the dome. Lastly, opening the dome altogether increases the volume with a respective signature of $*532$ but with a new pattern.

The structure consists of six pentagons and ten triangles interconnected by threads at their vertices. Each



(a) (b) (c) (d)

Figure 3: *Defining a dynamical geodesic dome.*

pentagon connects to five triangles, and each triangle connects to three pentagons. By interpreting the signs depicted, one can understand the model's dynamics, facilitating transitions between different states. The blue marks denote the vertices where threads are connected, forming a flexible mechanism indicated by the yellow marks. This allows vertices to move freely around each other. Movement begins when a shape, like the upper pentagon, rotates clockwise, causing connected triangles to rotate counterclockwise. This mechanism resembles the operation of gear wheels. This rotational movement initiates a chain reaction among the remaining pentagons and triangles, resulting in counterclockwise rotation of triangles and clockwise rotation of pentagons. This pattern arises from the clockwise rotation of pentagons and counterclockwise rotation of triangles within the model. Hence, the interconnected vertices create a simple yet efficient mechanism driving sequential movement when one part starts moving. This movement causes the dome's radius to fluctuate.

The three stages of domes are obtained manually and can be operated by a mechanical system similar to [1], which controls the pattern and the volume of the dome simultaneously.

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