Cosmatesque Geodesics

Steve Pomerantz

Brooklyn, New York, USA; circleofsteve.com; circleofsteve@gmail.com

Abstract

This article illustrates the construction of geodesics on surfaces of revolution embellished with traditional Cosmatesque or Byzantine motifs. On such surfaces geodesics can be found using a particularly simple relationship due to Alexis Claude de Clairaut.

Introduction

In geometry, a geodesic represents a unit-speed curve that is locally of shortest length, alternatively thought of as a generalization of "straight" lines on curved surfaces. The topic has a rich history and is detailed in many books on classical differential geometry and general relativity [1][2][5]. The theory leads to a system of second-order differential equations whose solution describes a 1-parameter curve, the geodesic. This paper presents an algorithm for drawing a pattern on a particular family of surfaces, namely surfaces of revolution, like a sphere and a torus; but also, could include other types of objects that are typically made with a lathe (e.g. a bat) or a potter's wheel (e.g. a bowl).

Construction

Typically, a surface of revolution is parameterized by two variables u and v, where u measures the rotation about the central axis, and v represents the parameterization of the profile curve. The horizontal circles described by v = constant are called parallels, while the rotated versions of the profile curve defined by u = constant are called meridians. In terms of u and v, if the profile curve is described as $(x, z) = (f(v), g(v))$ then the surface is described by $(f(v)\cos(u), f(v)\sin(u), g(v))$.

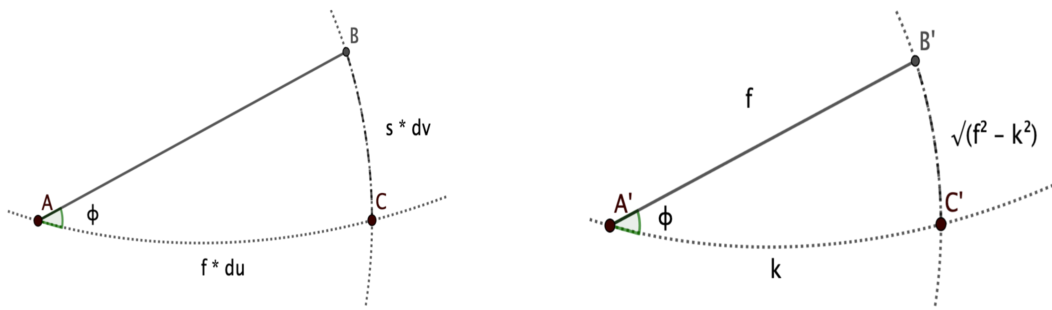
In the case of a surface of revolution the geodesic equations simplify significantly due to the Clairaut relation. This allows for a simpler approach to identifying geodesics than solving a system of differential equations. Along any curve the value of $f(v)$ represents the distance between a point and the axis of rotation. If ϕ represents the angle between the curve and a parallel through such point, then the Clairaut's observation is that the product of $f(v)$ and $\cos(\phi)$ remains constant, k, along the curve. As such, $\cos(\phi)$ and thus, ϕ, is forced to change as the distance from the axis changes.

As a result of this relationship, the values for u and v along the geodesics on a surface of revolution, must satisfy the formula below which can be stepwise integrated to produce the full curve:

$$dv/du = \frac{\left(\frac{f}{k}\right) * \sqrt{f^2 - k^2}}{\sqrt{f'^2 + g'^2}}$$

Intuitively, this equation derives from equating $\tan(\phi)$ in both of the triangles below. The hypotenuse in Figure 1a represents the tangent to the geodesic. AC represents a segment of a latitude, and BC a segment of a meridian. The lengths of the shorter sides are given by their respective arc-lengths where s equals

$\sqrt{f'^2 + g'^2}$ is the line-element along the profile curve. The image on the right is a graphical depiction of the Clairaut relationship, *f(v)* * cos(ϕ) = *k*.

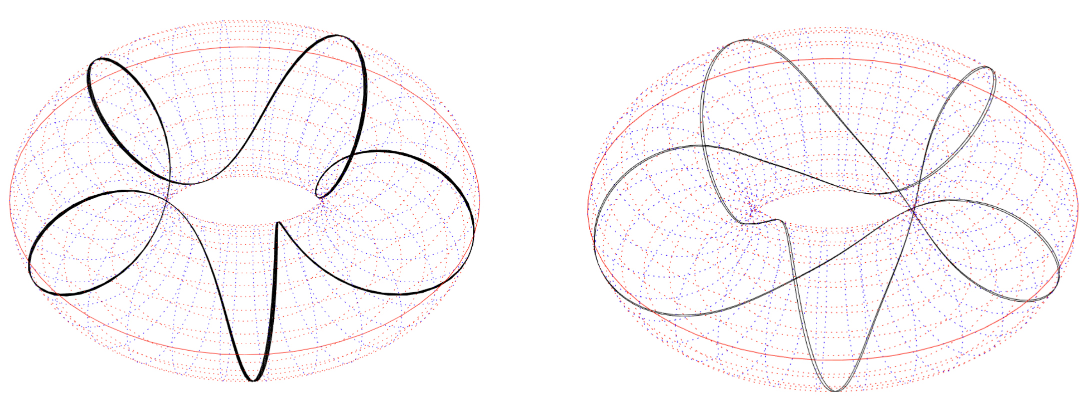


(a) (b)

Figure 1: *Infinitesimal triangles used to define the tangent to the geodesic: (a) based on meridian and parallel (b) based on Clairaut*

The types of curves generated will depend on the value of *k* chosen. On the sphere, the solutions are just the great circles, which are circles that are the intersection of the sphere with a plane that passes through the center.

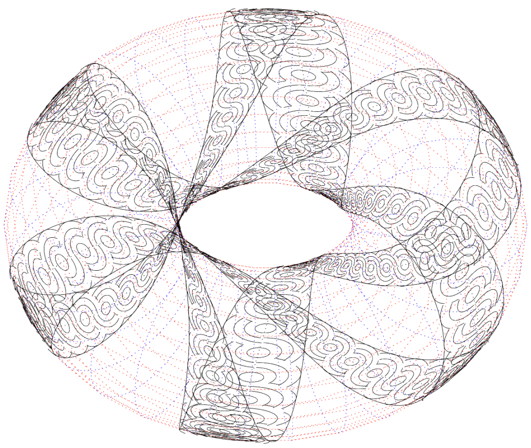
A more interesting example comes from examining the torus, where f(v) = a + b cos(v) and g(v) = b sin(v). In this case the relationship between k, a, and b can lead to very different behaviors. Given that cos(ϕ) is bounded by 1, the chosen value of k can place limits on how close (or far) the geodesic can get to the central axis. Excellent descriptions and details are provided in [3][4] that illustrate solutions for different ranges of k. By selecting k appropriately, the geodesic can be made to close, that is return back to the starting point, oriented in the same direction. Furthermore, for certain values of k, the curve can be made to go around both the axis of rotation as well as the equator an arbitrary number of times. Both of the examples below in Figure 2, represent curves that go around the equator 5 times. The curve on the left, Figure 2(a) only goes around the central axis once, while the second curve in Figure 2(b) makes 2 revolutions before closing back up.



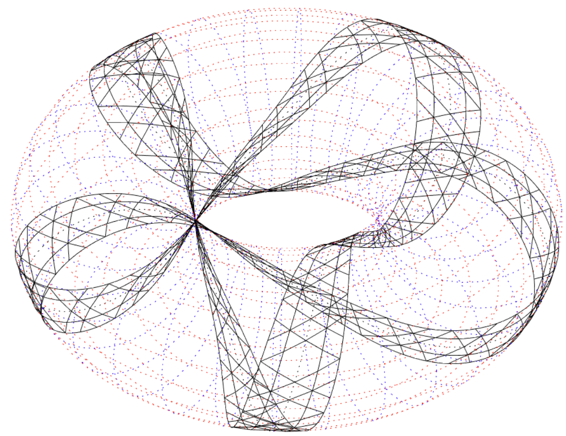
(a) (b)

Figure 2: *Examples of geodesics on a torus: (a) drawn as a pentagon with 1 rotation around the central axis and (b) as a pentagram requiring 2 rotations around the central axis. All graphics are developed in MATLAB®*

The initial geodesic is then found by using a suitable finite difference algorithm, like Euler or Runge-Kutta. Once found, a set of similar geodesics can be found by rotating the initial curve around the central axis to generate one set of lines that will be part of a grid on the surface. This generates identical geodesics that cross the equator for different values of u. Grids constructed between these curves can now form the basis for tessellations or other patterns. In the examples below, the curves on the left, Figure 3(a), are filled with a traditional Guilloche pattern and Figure 3(b) on the right with a traditional tiling of hexagons, rhombi and triangles. These patterns and many other traditional Cosmatesque designs are illustrated in [6].



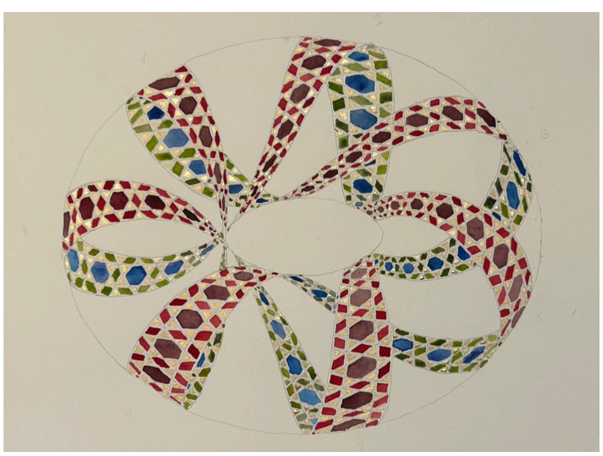
(a)



(b)

Figure 3: *(a) two pairs of geodesics each going around the equator 3 times, with a traditional guilloche, (b) a pair of geodesics internally tiled with polygons. Note that the boundaries of the strip only appear to intersect due to this particular perspective.*

The following paintings represent a sample of possible patterns made using this method.



(a)

(b)

Figure 4: *(a) a pair of curves going around the equator 7 times and (b) two pairs of geodesics each going around the equator 5 times.*

The final examples, Figure 5, are based on using a different profile curve than a circle and illustrate the variety of this approach. The figure on the left rotates a wave curve given by $z = \sin(x) + 1$ and the figure on the right rotates a bell curve given by $z = e^{(-x^2)}$. Note how the curve on the right is forced to turn around given that it is prevented from getting any closer to the central axis.

(a) (b)

Figure 5: *(a) profile curve given by sin(x) + 1, (b) profile curve given by* $e^{(-x^2)}$

References

[1] M. do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976

[2] S. Carroll. *An Introduction to General Relativity Spacetime and Geometry*. Cambridge University Press, 2019.

[3] V. Coll and J. Dodd. "Why Curves Curve: The Geodesics on the Torus." *Mathematics Magazine*, May 2022.

[4] M. Irons.*The Curvature and Geodesics of the Torus*. http://www.rdrop.com/%7Ehalf/math/torus/index.xhtml

[5] J. Oprea. *Differential Geometry and Its Applications*. Second Ed., The Mathematical Association of America, 2007

[6] P. Pajres-Ayuela. *Cosmatesque Ornament: Flat Polychrome Geometric Patterns in Architecture*. W.W. Norton & Co. 2001