Handle-Bodies Inspired by Tengstrand's "3-2-1"-Sculpture

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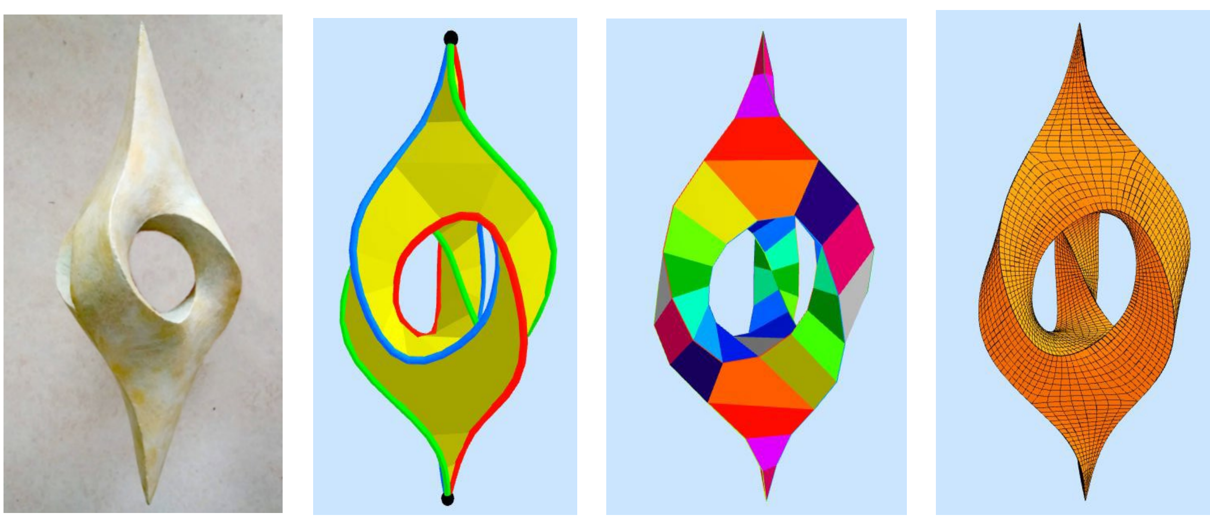
Abstract

This is an analysis of the geometry of the "3-2-1"-sculpture by Tord Tengstrand, presented at the Bridges 2020 Art Exhibition. By changing some of the geometrical parameters, derivative shapes of higher complexity are generated.

Introduction

In the Bridges 2020 Art Exhibition, Tord Tengstrand [6] presented an intriguing sculpture (Fig.1a) called "3-2-1", since it had 3 curved edges, 2 "pointy" vertices, but only a single, higly curved "face" that is topologically equivalent to a disk with multiple holes. This surface is composed of three ribbon-like arms that connect two 3-way junction areas. The three arms border each other along the three "loopy" edges that connect the two vertices. This defines the orientable boundary of a handlebody of genus 2, i.e., a 2-hole torus or, equivalently, the thick shell of a hollow sphere with three "tunnels" or "windows" to the interior. From a single image it is difficult to understand this shape. In a lively email exchange, Tord helped me to understand this shape and then find ways to extend this geometry to more complicated shapes.



(a) (b) (c) (d)

Figure 1: *"3-2-1"-Sculptures: (a) Tengstrand's original model; (b) enhanced CAD model showing the two vertices and the three edges; (c) a polygonal starting geometry, (d) smoothed by subdivision.*

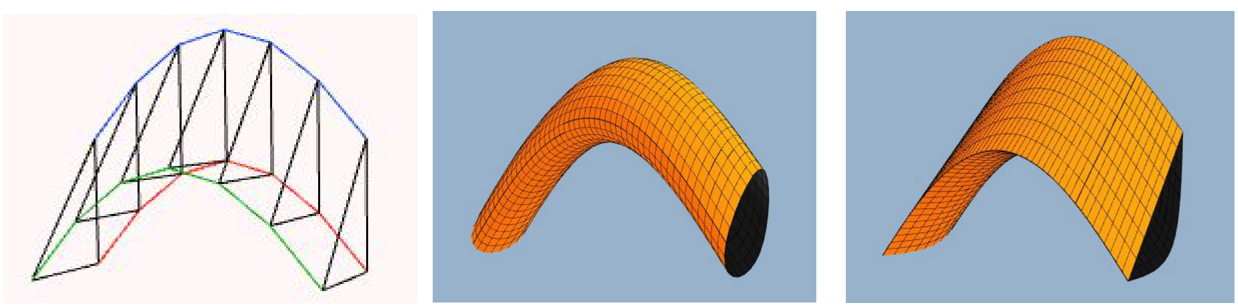
The Geometry of Tord Tengstrand's "3-2-1"-Sculpture

Tord's "3-2-1"-sculpture is a handlebody of genus 2 with 3-fold rotational D_3 symmetry around an axis that passes through the two (black) vertices in Figure 1(b). Three loopy, sharp edges with a dihedral angle of about 60 degrees, shown in red, green, and blue in Figure 1(b), are running between the two vertices, while partially circling around two of the three windows to the interior. The whole surface of this shape consists of a single, complicated (yellow) "face" that is smoothly connected.

To create a CAD model of this shape, I used our "homebrewed" JIPCAD environment [4]. This is a design tool that has evolved over more than two decades [5], [2], [3]. It combines precise, parameterized, procedurally specified geometry with an interactive graphical interface, through which users can make modifications and refinements. The main challenge in this CAD tool is to integrate any changes made interactively on the graphics screen back into the initial JIPCAD program without destroying its logical, hierarchical structure, and while also keeping fully functional all adjustable parameters in the initial program. This environment seems ideal for the design of modular geometrical sculptures with a high degree of symmetry.

I started with a definition of one half of one loopy edge specified as a cubic B-spline with only nine control points. Six properly rotated copies of this curve are then combined into the complete network of the three edges between the two vertices. About one sixth of the surface is now defined with a set of a dozen quadrilaterals, connecting two point-pairs on two neighboring branches of the loopy edges. This mesh is then instantiated six times to form the overall shape with D_3 symmetry. Overall, three more quads were needed to connect the upper half of the sculpture to the lower half; and in the center of each half, a (gray) triangular facet is included to form one of two 3-way junction areas for the complicated, multiply connected "face." This leads to a polyhedral shape (Fig.1c), which can then be smoothed (Fig.1d) by three levels of Catmull-Clark subdivision [1].

However, just subjecting the initial polygonal model to a few levels of Catmull-Clark subdivision does not capture the spirit of Tord's "3-2-1"-sculpture; it lacks the clearly defined, sharp edges (Fig.1a) that are a key element of this type of sculpture. Figure 2(b) shows what happens to a 3-sided prismatic sweep (Fig.2a) when plain CC-subdivision is applied: The result is a smooth tube! However, sharp edges can be preserved (Fig.2c) if they are marked explicitly as "sharp" in the polyhedral model, i.e., the top (blue) and bottom (green) edges in Figure 2(a). The CC-subdivision process will then avoid any smoothing across such marked edges, and the desired model with sharp edges can be obtained (Fig.1d).



(a) (b) (c)

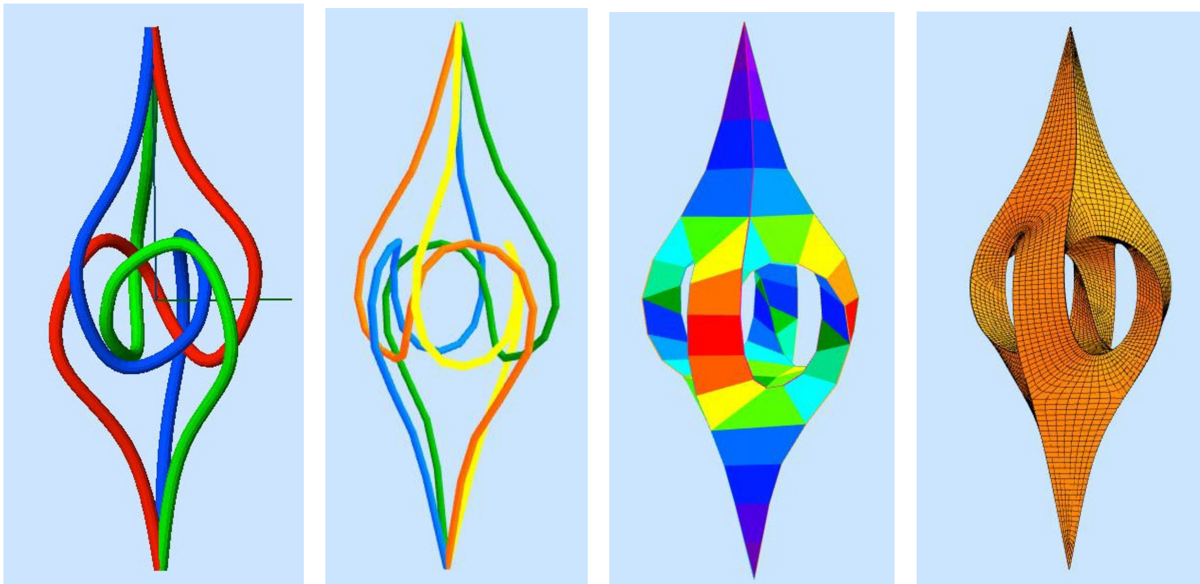
Figure 2: *(a) Prismatic polyhedron: (b) Plain CC-subdivision applied. (c) Two edges marked "sharp."*

Increasing the Genus by Going to "*N*-2-1"-Geometries

This intriguing sculpture raised the question whether similar sculptures could be made with more prismatic "handles" between the two vertices, with higher symmetry and correspondingly higher genus. Ideally, all edges have the same shape, and overall the handlebody maintains a $D_{N\text{-}1}$ symmetry.

A "4-2-1"-Handlebody

For my first derivative shape, I increased the genus of the original sculpture by enhancing the rotational symmetry of the set of edges from D_3 (Fig.3a) to D_4 (Fig.3b). One eighth of the new multi-arm face of this sculpture is constructed in the same way as in the genus-2 version by suspending a manageable set of quads or of triangular facets between neighboring segments of different edge-curves (Fig.3c). This polyhedral model of genus 3 can then be smoothed by CC-subdivision that preserves explicitly marked curves as sharp edges in the final sculpture (Fig.3d).

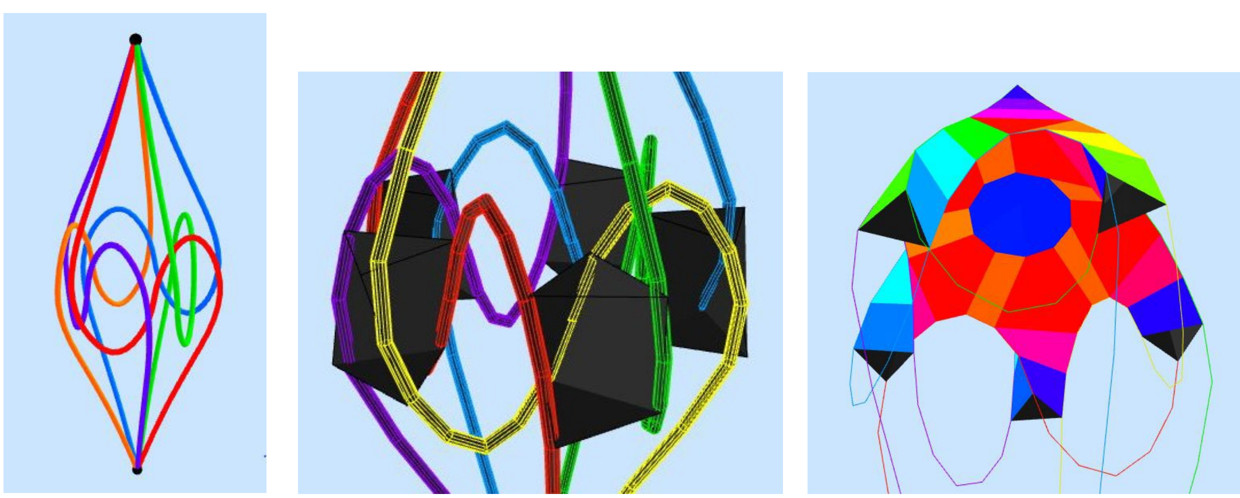


(a) (b) (c) (d)

Figure 3: *Increasing the rotational symmetry: (a) The 3 edges of the original "3-2-1"-sculpture; (b) four edges for an enhanced "4-2-1"-shape; (c) connecting the four edges with small facets; (d) the resulting shape smoothed by CC-subdivision with sharp edges.*

A "5-2-1"-Handlebody

The same approach can be taken to make additional derivative shapes with higher symmetry and higher genus. Figure 4 shows the construction of a genus-4 shape with D_5 symmetry. Figure 4(a) shows the set of five loopy edge-curves connecting the two vertices. To insure that I will obtain some nice 3-sided solid branches between the two vertices, I have added some auxiliary scaffolding in the shape of five parameterized 3-sided (black) prisms (Fig.4b). Their positions, radius, twist, and azimuthal orientation can be adjusted interactively via on-screen sliders on the graphic screen. Once these prisms have been placed to render a pleasing-looking handlebody, the control polygons for the five B-spline edge-curves are adjusted to line up with the longitudinal edges of the scaffolding prisms. This produces the polyhedral geometry displayed in Figure 4(c). Figure 5 depicts the overall result for a "5-2-1"-handlebody.



(a) (b) (c)

Figure 4: *(a) A set of edges for an enhanced "5-2-1"-handlebody; (b) five auxiliary prism structures to facilitate the design of suitably curved edges; (c) half of the polyhedral surface model.*

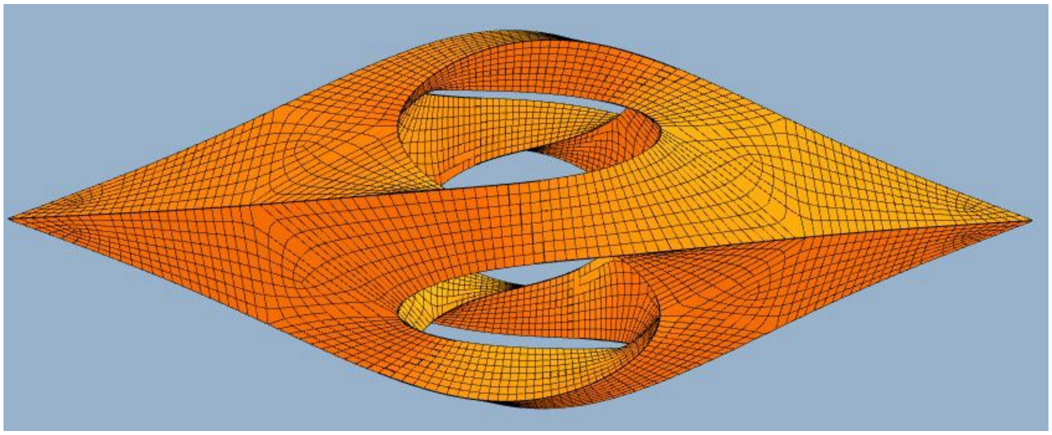
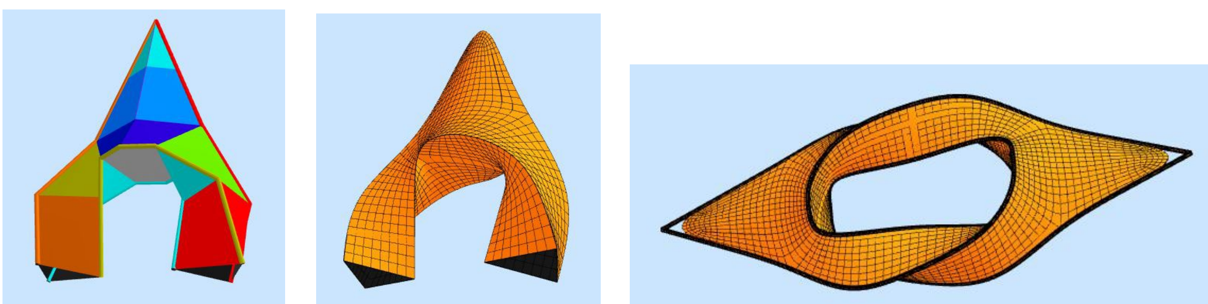


Figure 5: *Handlebody "5-2-1" smoothed by CC-subdivision with sharp edges.*

A "2-2-1"-Handlebody

For completeness, I also constructed a "2-2-1" handlebody of genus 1. I started with a coarse polyhedral model (Fig.6a) of just half the envisioned 2-branch handlebody. Even with this rather crude starting model, CC-subdivision does an amazingly good job of creating a pleasing shape (Fig.6b). I then connected two of these modules to obtain a handlebody that clearly belongs into this family of "*N*-2-1" shapes where all the edges make 3 passes past the central void on their zig-zag path from one vertex to the other.



(a) (b) (c)

Figure 6: *Handlebody "2-2-1": (a) polyhedral starting model; (b) smoothed by CC-subdivision, (c) and combined into a complete "2-2-1"-shape of genus 1.*

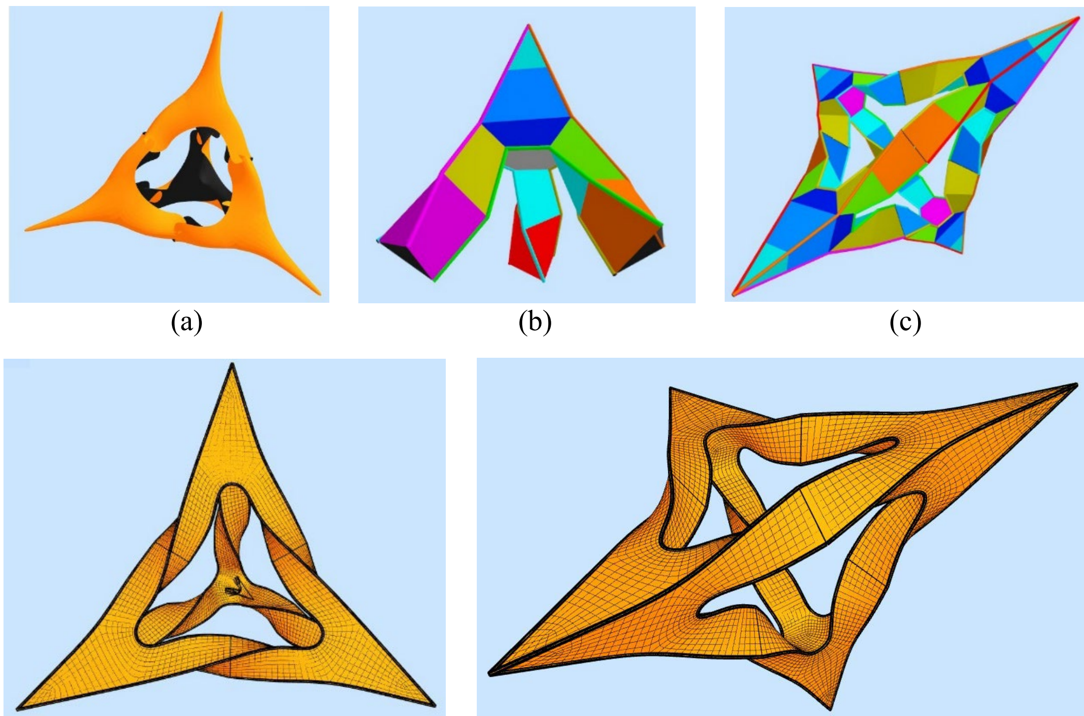
Composing Half-Sculptures into Higher-Genus Handlebodies

Another way to derive more complicated handlebodies from Tord's sculpture is to use just half of this geometry as a pliable, parameterized building module. Half of Tord's "3-2-1"-sculpture forms a 3-sided pyramid with three solid, 3-sided prismatic "legs" protruding from its bottom. These legs can be connected to the legs of other modules to form the edges of some Platonic polyhedral frame. Again, I aim for maximal symmetry, and I would like to have all the edge-curves to be of the exact same shape.

Tetrahedral Handlebody

In the first design, I placed four pyramid modules at the corners of a large tetrahedron and joined the twelve legs to form the six edges of the tetrahedral frame. Figure 7(a) illustrates the concept. It was easy to obtain the basic tetrahedral frame with six solid branches. But it was difficult to define the right kind of "loopy" edges that together form the borders of one or more multi-arm faces on the surface of the tetrahedral handlebody.

Encouraged by the success of the construction shown in Figure 6, I started with half the model shown in Figure 1(c). But I added a few more adjustable parameters. I kept the triangular prisms at the end of each leg adjustable in diameter, azimuth, and twist, and I added the freedom to adjust the tilt and the separation of these prisms to spread the three legs of the pyramid more broadly (Fig.7b). This allows me to obtain a smooth connection into a tetrahedral assembly (Fig.7c). Specifically, to obtain a flush joint between the legs of adjacent pyramids, the tilt of the prisms against the symmetry axis of the pyramid must be 35.26 degrees. It then turns out that each one of the six individual edges follows a contorted path that starts at one pyramid vertex, passes on the inside of two other pyramid vertices, and then ends on the fourth vertex. I would not have been able to design such a path based on a B-spline.



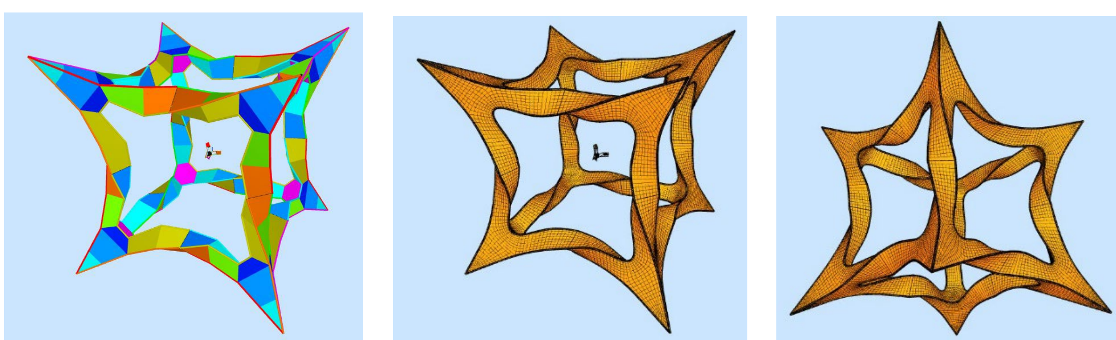
(a) (b) (c)

(d) (e)

Figure 7: *(a) Tetrahedral handlebody. (b) Half of a "3-2-1"- shape forms a "pyramid" with three legs. (c) Four pyramids joined into a tetrahedral frame. (d,e) Two views of the resulting "6-4-1"-tetrahedron.*

Cuboidal Handlebody

In the next experiment I combined eight of the above parameterized pyramid modules in to cuboidal shape. Figure 8 shows the construction and the network of the twelve loopy edges.

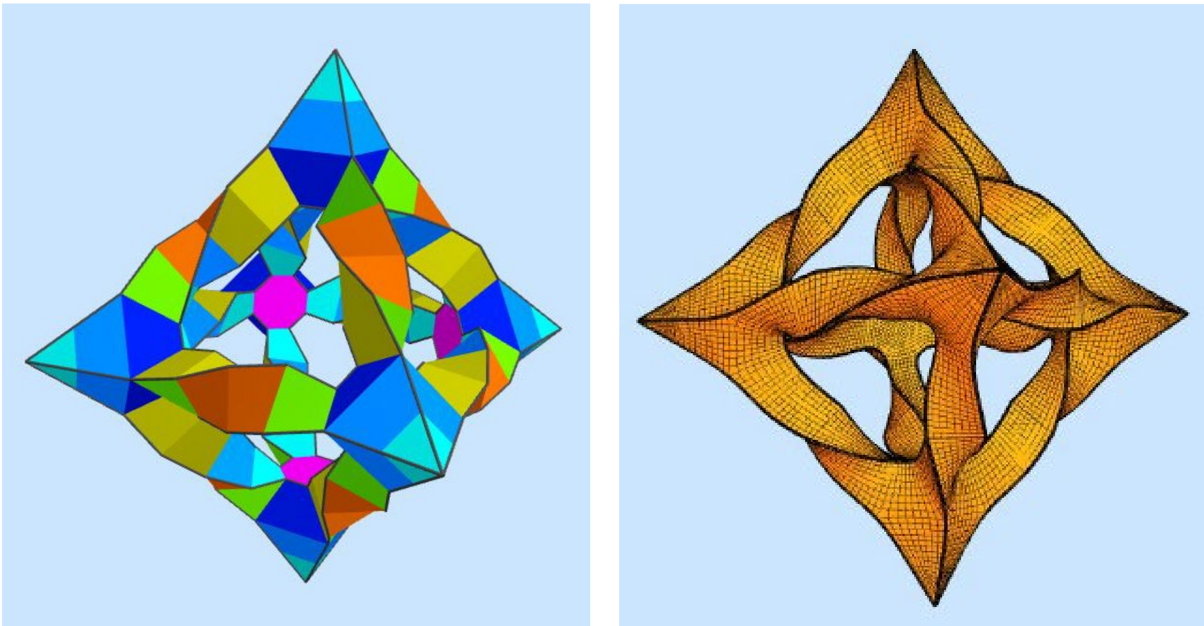


(a) (b) (c)

Figure 8: *(a) Eight pyramids forming a cuboid framework. (b,c) Resulting structure after subdivision.*

Octahedral Handlebody

Next, I used the same basic approach to combine six 4-legged pyramid modules into an octahedral configuration. Again, each edge passes through three of the twelve branches of this handlebody.

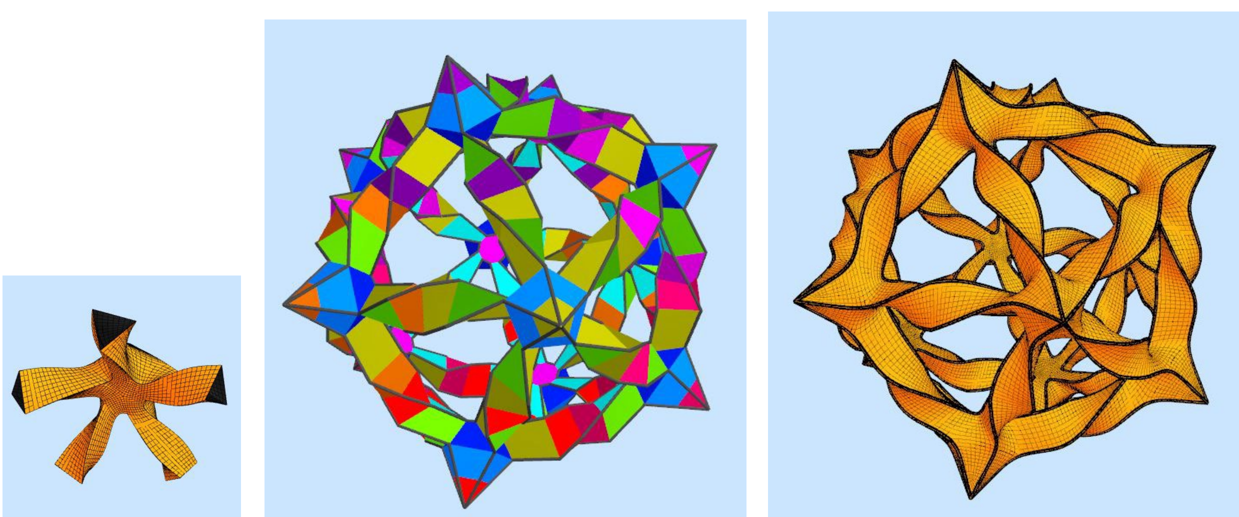


(a) (b)

Figure 9: *(a) Six 4-leg pyramids forming an octahedral framework. (b) Resulting structure after CC-subdivision with the sharp edges highlighted.*

Icosahedral Handlebody

Twelve copies of a 5-legged pyramid can be assembled to form an icosahedral structure. Now the legs of the pyramid module are spread much wider (Fig.10a) to obtain an overall icosahedral shape (Fig.10b,c).



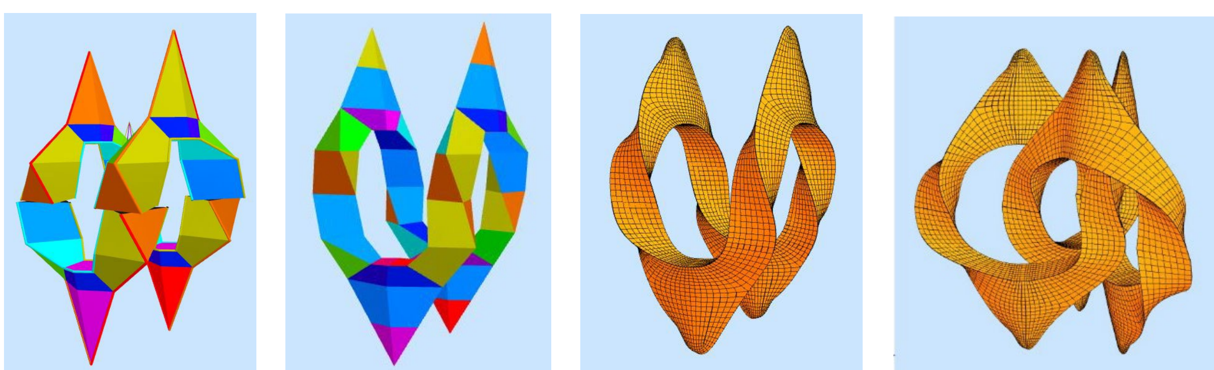
(a) (b) (c)

Figure 10: *(a) The re-shaped 5-legged pyramid corresponding to half of an "5-2-1"-shape. (b) Twelve pyramids joined in an icosahedral framework. (c) Resulting handlebody after CC-subdivision.*

Other Assemblies, and Conclusions

The last four models have covered four of the five Platonic solids. Clearly one could also assemble twenty 3-leg pyramids into a dodecahedral framework. By slightly deforming the pyramid module and combining pyramid modules with a different number of legs, one can also create frame structures of the Archimedean solids. Moreover, any arbitrary polyhedron could be turned into a corresponding frame structure by placing different pyramids with the right "valence" at every vertex of the original polyhedron. It appears that maintaining the original edge-structure of the pyramid modules shown in the previous sections will also result in a set of sharp edges winding around the solid branches of the frame structure in a way that creates a single, multi-arm face of the Tengstrand kind. However, many of those structures look too "airy" and may not convey the intriguing compactness of Tord's original "3-2-1"-sculpture. Therefore, I have started to investigate other ways of making handlebodies in "Tengstrand-style".

As a starting point, I explored what I could do with just a few copies of the 2-leg pyramid module shown in Figure 6(a). With only two legs, it cannot form a true polyhedral network. But I can form a closed-loop strand. Using just three units would lead to a star-shaped triangle; but this seems rather flat and uninteresting. But four modules can be connected into an "up-down-up-down" zig-zag path that results in a more compact, 3-dimensional configuration. But I ran into an interesting difficulty. While two "Half-2-2-1"-modules (Fig.6a) readily assemble into a "2-2-1"-shape (Fig.6c), I could not readily connect all four 3-sided leg pairs in the anticipated zig-zag structure to form flush, lined-up connections (Fig.11a). I had to enhance the parametrization for this model to allow me to independently azimuth-adjust the two 3-sided prisms at the ends of the two legs. This broke the symmetry of the "Half-2-2-1"-module, and I had to construct two independent, slightly different polyhedral surfaces for the front and back of this module. This then allowed me to have flush connections in all four legs (Fig.11b) without even having to change the selection of a set of sharp edges, highlighted in Figure 11(a). CC-subdivision then yields the handlebody of genus 1 (Fig.11c) that displays the typical characteristics of the simpler Tengstrand shapes discussed in the first half of this paper. On the other hand, because the legs are 3-sided, six of these modules can readily form a zig-zag assembly with a hexagonal footprint. They can even be interlinked into a chain that forms a trefoil knot (Fig.11d).

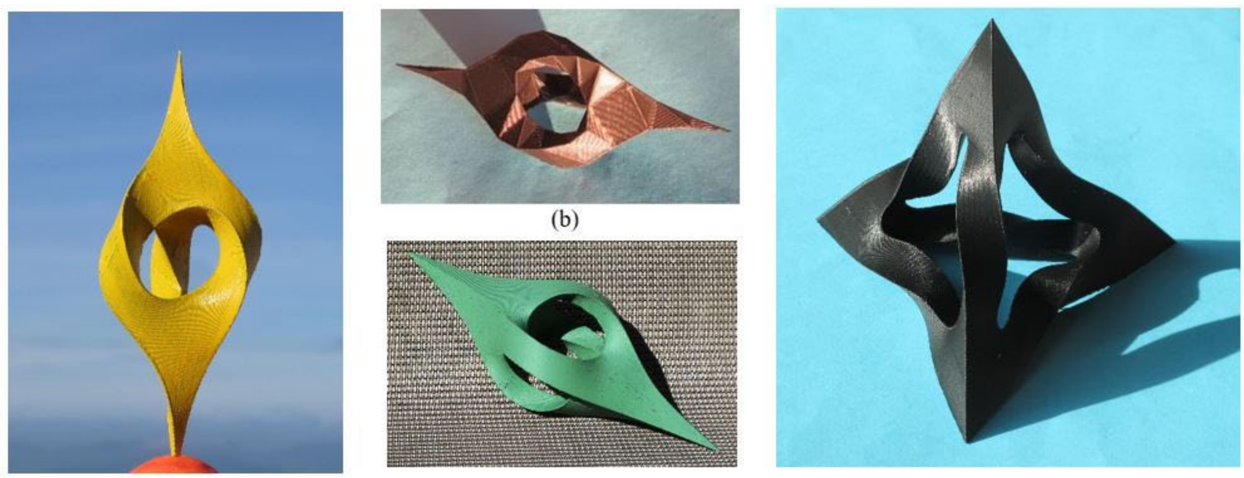


(a) (b) (c) (d)

Figure 11: *(a) Four 2-leg pyramids in a zig-zag loop with mis-aligned joints. (b) After modifying the parametrization to allow different azimuth adjustments in the two legs of the module, all joints can be aligned properly. (c) Resulting structure after CC-subdivision. (d) Six 2-leg pyramids interlinked to form a trefoil knot.*

3D-Print Models

Recently I started to fabricate some of these shapes on low-end 3D-printers. The somewhat "dangerous" looking forms cause no hard problems on the Ultimaker printers [7]. The only adjustments I had to make to the default settings was to reduce the overhang-angle from 60 degrees to 40 degrees for faces that need no direct support material below them. This then allowed to reliably construct wedges with rather narrow dihedral angles pointing straight downwards. Figure 12 shows some sample prints. First, I re-modeled Tord Tengstrand's original "3-2-1"-sculpture. Figures 12(a) and 12(b) show both the desired smooth shape and the polyhedral model that turns into this shape when CC-subdivision with sharp edges is employed. Figure 12(c) shows a derivative "4-2-1"-model of genus 3 described in Figure 3. Figure 12(d) shows the tetrahedral assembly discussed in Figure 7. It is advisable to print these objects in a light color, to obtain good shading variations in any photographs. If a black filament is the only option, then it is worthwhile to paint the models with some bright color (Figs.12a,b,c).



(a) (c) (d)

Figure 12: *3D-prints: (a) Replica of Tord's "3-2-1"-sculpture; (b) its polyhedral starting shape. (c) First derivative shape: a smooth "4-2-1"-handlebody. (d) Tetrahedral assembly "6-4-1".*

All models depicted in this paper should be readily printable on low-end printers using a support structure that is manually removed. Two such models, "Octoid-12-6-1" (Fig.9) and "Hex-Tangle-6-6-1" (Fig.11d) have been submitted to the 2024 Bridges Art Exhibition.

Acknowledgment

I am grateful to Tord Tengstrand for presenting his "3-2-1"-sculpture at Bridges 2020, and for involving me in a lively email discussion about its geometry and its topology. This allowed me to do the fascinating explorations presented in this paper.

References

[1] E. Catmull, J. Clark. "Recursively generated B-spline surfaces on arbitrary topological meshes." – https://people.eecs.berkeley.edu/~sequin/CS284/PAPERS/CatmullClark_SDSurf.pdf

[2] G. Dieppedalle. "Interactive CAD Software for the Design of 2-manifold Free-form Surfaces." Tech-Report (EECS-2018-48). –https://www2.eecs.berkeley.edu/Pubs/TechRpts/2018/EECS-2018-48.pdf

[3] R. Fan, C. H. Séquin, R. Ng. "Joining Interactive Graphics and Procedural Modeling for Precise Free-Form Designs." Tech-Report (EECS-2021-125) May 14, 2021. – https://www2.eecs.berkeley.edu/Pubs/TechRpts/2021/EECS-2021-125.html

[4] JIPCAD Language Reference. – https://tinyurl.com/9ds62ee2

[5] J. Smith. "SLIDE design environment." (2003). – https://people.eecs.berkeley.edu/~ug/slide/

[6] Tord Tengstrand "3-2-1-Sculpture." Bridges 2020 Art Exhibit – https://gallery.bridgesmathart.org/exhibitions/2020-bridges-conference/tordtengstrandteliacom

[7] Ultimaker. "Reliable 3D printers." – https://ultimaker.com/3d-printers