Frieze Decompositions of Wallpaper Patterns: Origami Models for the cmm Class

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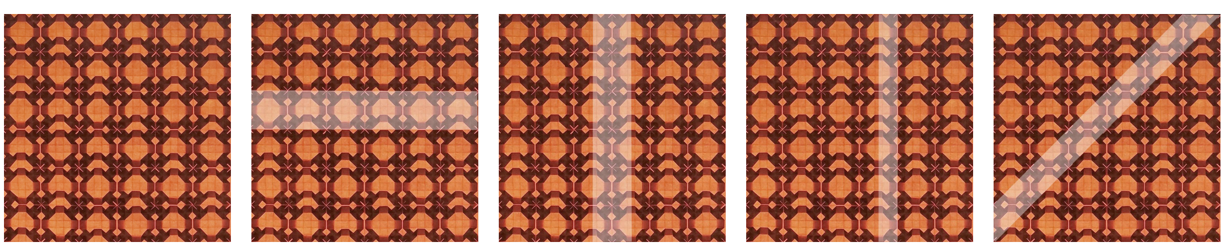
Abstract

This paper presents six subtypes of the wallpaper pattern class *cmm*, based on the frieze pattern types of the strips delineated by adjacent reflection and glide axes in two orthogonal directions. We present realizations of all six subtypes as origami models that can be physically interchanged through local pleat adjustments.

Introduction and Motivation

The theme of this article is the interpretation of wallpaper patterns as unions of parallel friezes of equal width, with disjoint interiors, and the use of origami to explore and exhibit such representations. Figure 1(a) below shows a backlit photograph of an origami model, created by the author, of a periodic plane pattern from the wallpaper class *cm*.

Each of the additional images in Figure 1(b-e) shows the same photograph, with a particular strip highlighted. Each of the four highlighted strips extends to a frieze within the pattern, and together they represent four of the seven distinct frieze pattern classes. In each case, we may consider the entire *cm* pattern to be a union of strips parallel to the highlighted one, where all have the same width, adjacent strips share a boundary line, and distinct strips have disjoint interiors. A similar construction may be applied to any frieze that occurs as a strip within any wallpaper pattern. We refer to the resulting arrangement as a *frieze decomposition* of the pattern, provided that every strip in the union is a frieze (which is not automatic). As long as the translation group of the pattern includes a vector parallel to the edges of the strip (to ensure periodicity), and the strip width is sufficient that each strip in the union captures enough features of the wallpaper pattern to induce a frieze pattern, the arrangement of parallel strips is a frieze decomposition. Properties of the examples in Figure 1(b-e) are discussed in more detail in the next section.



(a) **(b)** **(c)** **(d)** **(e)**

Figure 1: *cm pattern with highlighted friezes*

A periodic strip within a wallpaper pattern belongs to one of the seven frieze classes, according to its symmetries. One important point is that the frieze symmetry group of a strip need not be a subgroup of the wallpaper group of the ambient pattern. For example, the symmetries of the strip in Figure 1(e) include reflections, glide relections and 180° rotations that are not symmetries of the *cm* pattern. On the other hand, the translations of this strip are elements of the *cm* wallpaper group. The interplay of local and global symmetry elements is a general theme underlying the discussion throughout this article.

There are many very general questions that can be proposed about frieze decompositions. Given a particular frieze decomposition, one can note the sequence of frieze pattern classes represented by its successive parallel strips. For each of the seventeen wallpaper classes, one can try to identify which sequences of this kind must occur for every pattern in the class, or might occur for some of them. One can ask to what extent the answers to these questions distinguish or relate different wallpaper classes. The examples of friezes in Figure 1 already suggest that these questions may be too broad to admit very precise or meaningful answers, in view of the scope of available choices for the orientation and width of friezes in a decomposition. We can be more specific by considering friezes delineated by reflection and/or glide reflection axes of the pattern. In this article, we apply this idea to the *cmm* wallpaper class, leading to the identification of six distinct subtypes of *cmm* patterns.

A motivation for the idea of studying frieze decompositions is the apparently wide diversity of visual appearances of patterns within a single wallpaper class. This diversity is arguably more pronounced for some classes than others, although it is a perceived phenomenon that depends on human subjectivity and on the extent of individual experience with the visual task of recognizing symmetries. It is not always apparent at a glance that two patterns represent the same wallpaper class. While the objects and mechanisms of mathematics are not adequate to explain the diversity of subjective human experiences, we hypothesize that some of this variation may be explained by purely mathematical considerations of frieze decompositions. In our last section, we illustrate this for the *cmm* class, which is partitioned into six subtypes by frieze decompositions determined by reflection and glide reflection axes. We demonstrate the six subtypes with interchangeable origami models.

For background on the frieze and wallpaper classifications of periodic patterns in one and two dimensions, and on their groups of symmetries, we refer to [1] or [2].

Frieze Decompositions of Wallpaper Patterns

We begin by listing the seven frieze pattern types in Table 1, denoted here using the crystallographic convention. We consider a frieze to be oriented so that its infinite edges are horizontal. A reflection whose axis is parallel to these edges is called *horizontal*, and a reflection whose axis is orthogonal to the infinite edges is called *vertical*.

Table 1: *The Seven Frieze Types*

Class	Example	Symmetries of the frieze group
11	LLLLLLLLLL	translations only
1*m*	CCCCCCCCCC	translations, horizontal reflection and glide reflections
*m*1	VVVVVVVVVV	translations and vertical reflections
12	SSSSSSSSSS	translations and 180° rotations
mm	HHHHHHHHHH	translations, 180° rotations, horizontal reflection, vertical reflections and glide reflections
mg	VΛVΛVΛVΛVΛ	translations, 180° rotations, vertical reflections and glide reflections
1*g*	bpbpbpbpbp	translations and glide reflections

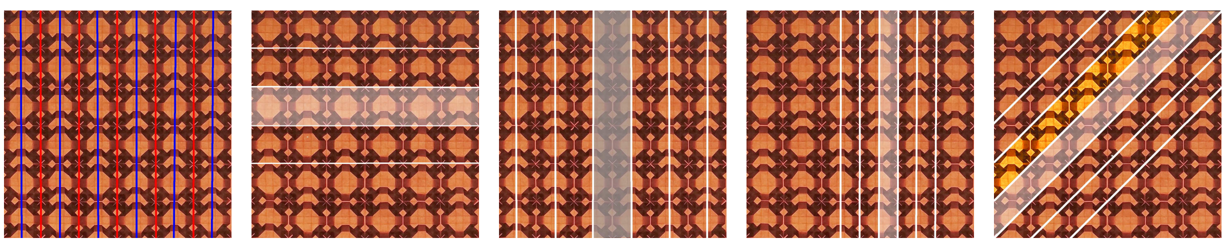
We now consider the frieze decompositions of the pattern in Figure 1(a) that are determined by the strips highlighted in Figure 1(b-e). All are redisplayed in Figure 2, with additional visual emphasis on the associated decompositions of the *cm* pattern. For reference, Figure 2(a) shows the reflection and glide axes of the pattern, respectively coloured blue and red.

The highlighted strip in Figure 1(b) belongs to the frieze class $m1$, and its vertical reflection axes all extend to reflection axes of the wallpaper pattern. Its translations are also translations of the wallpaper pattern, so its frieze symmetry group is naturally identified with a subgroup of the wallpaper group of the whole pattern. Some of its parallel shifts are outlined in Figure 2(b). They all belong to the class $m1$, and all are images of each other under translations of the wallpaper pattern.

In Figure 1(c), the highlighted strip is a frieze of type $1g$, and its boundary lines are reflection axes of the pattern. Figure 2(c) shows the frieze decomposition determined by this strip. All of its friezes have type $1g$, and all are images of each other under translations of the *cm* pattern.

In Figure 1(d), the highlighted strip is a frieze of type 11. Figure 2(d) shows more strips in its associated decomposition. All have type 11, and all are images of the highlighted one under elements of the *cm* wallpaper group, including compositions of reflections and translations. The boundary lines of each strip are a reflection axis and a glide reflection axis of the wallpaper pattern.

The example in Figure 2(e) is an interesting case. The highlighted strip in Figure 1(e) belongs to the frieze pattern class mg. Its reflections, glide reflections, and 180° rotational symmetries are local to this frieze and do not extend to symmetries of the wallpaper pattern. The strip above it, highlighted in yellow in Figure 2(e), has frieze type 11 and its translations are translations of the entire pattern. The edges of strips in this decomposition are not parallel or orthogonal to any reflection or glide axis of the wallpaper pattern. Successive friezes in the decomposition alternate between the classes mg and 11.



(a) **(b)** **(c)** **(d)** **(e)**

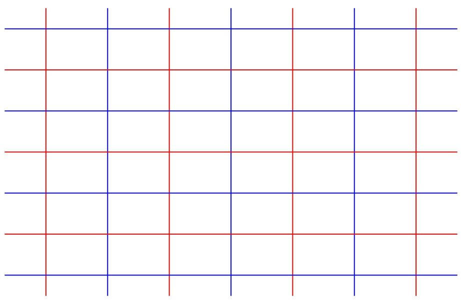
Figure 2: *Frieze decompositions of a cm pattern*

The examples in Figure 2 indicate some diversity in properties of frieze decompositions and in the relationships between the wallpaper group of the pattern and the frieze groups of strips, even within a single wallpaper pattern. Rather than attempt an analysis of frieze decompositions in full generality, we now direct attention towards some cases that are distinguished by symmetry considerations.

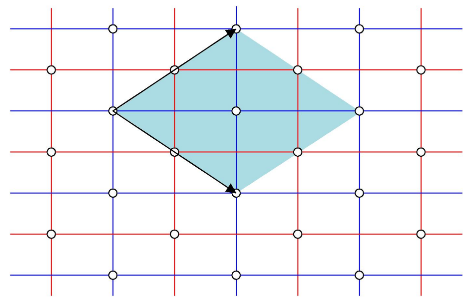
In wallpaper patterns that possess reflection and/or glide reflection symmetries, their symmetry axes are specified lines in the pattern that occur in parallel families. The strip between two adjacent members of such a family is always a frieze, and the collection of all such parallel strips is a frieze decomposition that is intrinsic to the symmetry of the wallpaper pattern. In this frieze decomposition, all of the strips belong to the same frieze class, being images of each other under the action of the wallpaper group. We say that a frieze decomposition is *intrinsic* if each of the two edges of each of its strips is either a reflection axis or a glide reflection axis of the wallpaper pattern, and no line parallel to these in the interior of the strip is a reflection or glide axis of the wallpaper pattern. In Figure 2, only the decomposition in (d) is intrinsic. It is possible that the centre line of a strip in a intrinsic frieze decomposition could be a reflection or glide reflection axis of the strip, but not of the wallpaper pattern.

The Wallpaper Class cmm

A *cmm* pattern has reflection axes in two perpendicular directions, with glide reflection axes in the same two directions, located midway between pairs of adjacent reflection axes. The *cmm* pattern class is not the only one that features this arrangement of reflection and glide axes, but it is distinguished by the fact that the reflections and glides in these axes generate the whole symmetry group. Figure 3(a) shows the arrangement of reflection axes (coloured blue) and glide axes (coloured red) in a typical *cmm* pattern. Figure 3(b) additionally shows centres of 2-fold rotations, a spanning pair of vectors for the lattice of translations, and a translation unit cell. A *cmm* pattern has no rotations of order exceeding 2, and its 2-fold rotation centres occur at the intersection points of reflection axes and at the intersection points of glide axes.



(a) *cmm reflection axes (blue) and glide reflection axes (red)*



(b) *cmm translation vectors, 2-fold rotation centres, and rhombic translation unit cell*

Figure 3: *Symmetries of a cmm pattern*

A familiar representative of the *cmm* wallpaper class is the "running bond" pattern that is the most common arrangement of bricks in a wall.

Intrinsic Frieze Decompositions in cmm: the Six Subtypes

Every *cmm* wallpaper pattern is equipped with two intrinsic frieze decompositions, one for each of its two families of parallel reflection and glide axes. Within each of these decompositions, all friezes are images of each other under translations, reflections, or glide reflections in the wallpaper group, so all belong to the same frieze class. The symmetry group of each frieze in the decomposition includes those vertical reflections that are symmetries of the entire wallpaper pattern. Therefore, the only possibilities for the frieze pattern class are the three that include vertical reflections: $m1$, mm and mg. Figure 4 shows the symmetries of a *cmm* pattern, with the position of one intrinsic frieze in each direction highlighted.

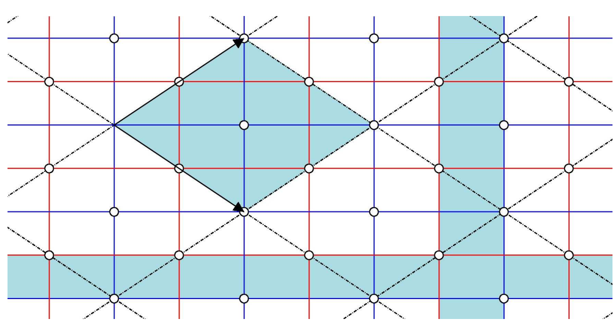


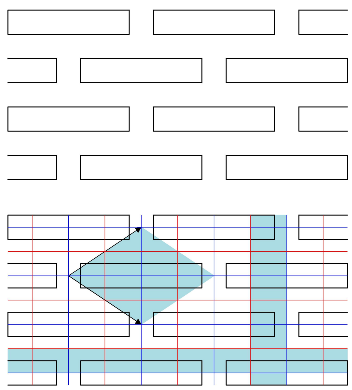
Figure 4: *Intrinsic frieze decompositions of a cmm pattern*

Given a *cmm* pattern, each of its two intrinsic frieze decompositions is associated to one of three frieze pattern classes. The pattern is thus associated to a pair of (not necessarily distinct) frieze pattern classes from these three. Every *cmm* pattern belongs to one of the six subtypes listed in Table 2, according to which pair of frieze classes occurs in its intrinsic frieze decompositions.

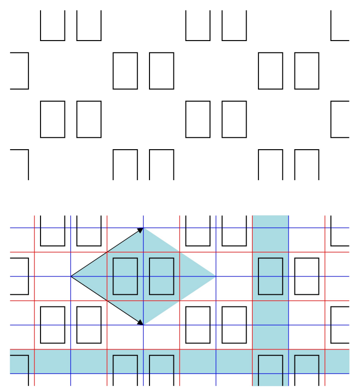
Table 2: *The Six cmm Subtypes*

Type 1: $m1$ and $m1$	Type 2: $m1$ and mm	Type 3: $m1$ and mg
Type 4: mm and mm	Type 5: mm and mg	Type 6: mg and mg

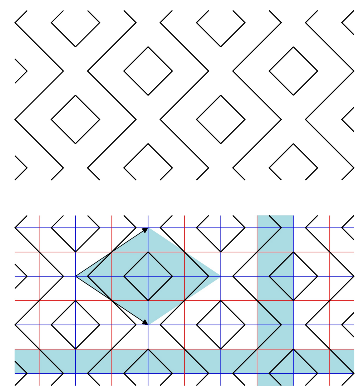
To determine the type of a particular *cmm* pattern, we identify the reflection and glide reflection axes and inspect an intrinisic frieze in each of the two orthogonal orientations to determine the relevant pair of frieze classes. That all six types occur is confirmed by Figure 5. One might consider Type 1 to be the “expected” or “default” type, since its friezes have only those symmetries that are enforced by the ambient *cmm* pattern. However, as Figure 5 shows, there is no extreme contrivance required to exhibit examples of all six types.



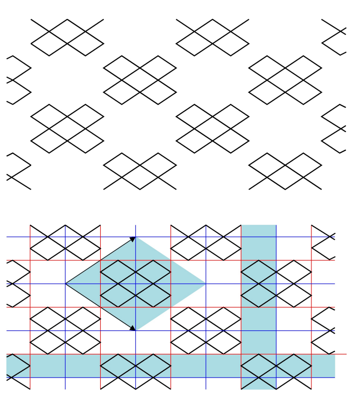
(a) *A cmm pattern of Type 1*



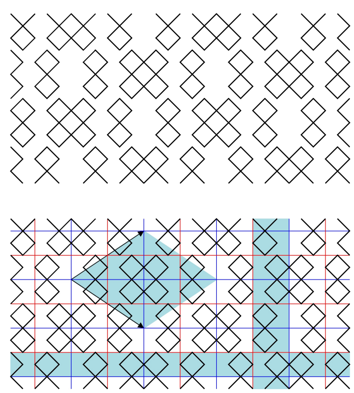
(b) *A cmm pattern of Type 2*



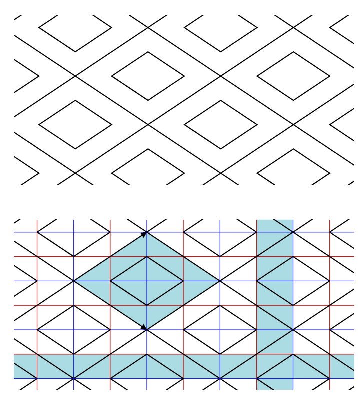
(c) *A cmm pattern of Type 3*



(d) *A cmm pattern of Type 4*



(e) *A cmm pattern of Type 5*



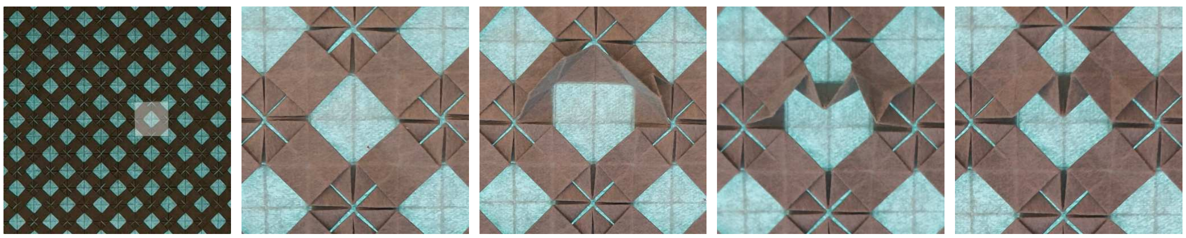
(f) *A cmm pattern of Type 6*

Figure 5: *Representatives of the six cmm subtypes*

Adaptable Origami Models for the Six Subtypes

We conclude with photographs of origami models of each of the six *cmm* subtypes, created by the author using locally adaptable designs. Each model is shown firstly without decoration, and secondly with highlighting of a rhombic translation unit, an intrinsic frieze in each direction, and reflection and glide axes respectively coloured blue and red. The second version may serve to confirm the subtype, as well as the *cmm* class itself.

All of the photographs in Figures 7-12 show the same 50cm×50cm sheet of Satogami paper, with patterns adjusted through local *pleat reorientations*. A pleat reorientation is a physical move that switches a single pleat in the model between left and right facing modes, like turning a page in a book. A pleat in an origami design generally intersects other features, and cannot be reoriented independently of its surroundings. The term *local* pleat reorientation does not have a precise definition; it refers to a move whose wider ramifications are confined to the immediate vicinity of the pleat. Figure 6(a) shows the starting point for all of the *cmm* models shown in this section; it is constructed by folding 196 square twists in a precreased square grid and then untwisting them. The images in Figure 6(b-e) show some steps in a local pleat reorientation of the only kind that is needed to obtain our models for the *cmm* Types 1 through 5.



(a) *initial model* **(b)** *starting point* **(c)** *lifting tabs* **(d)** *lifting pleat* **(e)** *repositioned*

Figure 6: *Local pleat reorientation*

Efforts to obtain a model of Type 6 using only pleat reorientations of this kind, while preserving the translation lattice, did not succeed. Our Type 6 model involves pleat reorientations of two kinds. In all of the models, the positions of the reflection and glide axes are preserved, which facilitates a comparison of the intrinsic friezes and translation unit cells.

The idea of incorporating locally reorientable pleats as a design consideration in origami models of periodic patterns, and using them to move from one wallpaper class to another, was proposed in the author's contribution to the Proceedings of the 2023 Bridges Conference [3]. Here the idea is extended to intrinsic frieze decompositions, with the *cmm* class as a test case. The basic design of Figure 6(a), equipped with the tool of pleat reorientation, provides a flexible mechanical framework that is extremely well suited to creating related models for the six *cmm* types and exploring their connections. That the orthogonal directions for the reflection and glide axes are embedded from the outset in the precreased grid is a matter of great practical convenience in the design and folding processes.

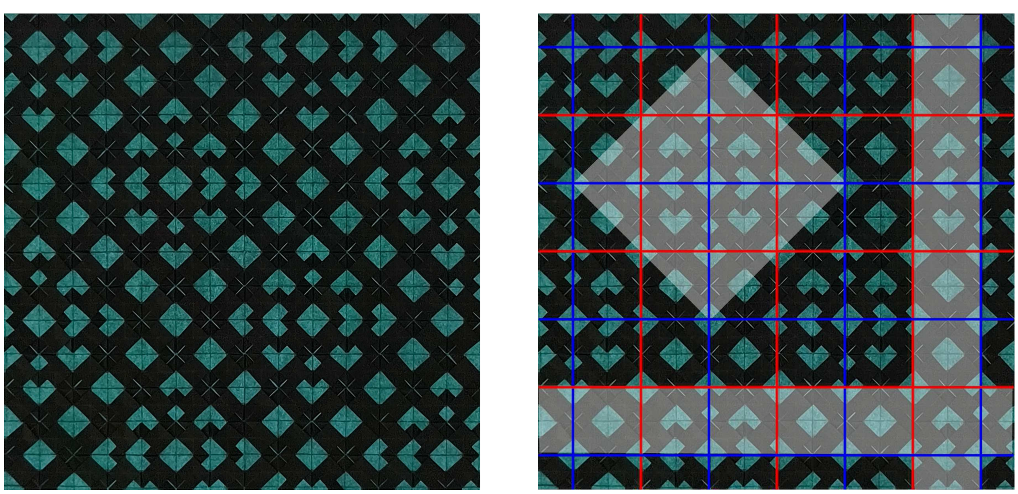


Figure 7: *Origami model of cmm Type 1 (m1 and m1)*

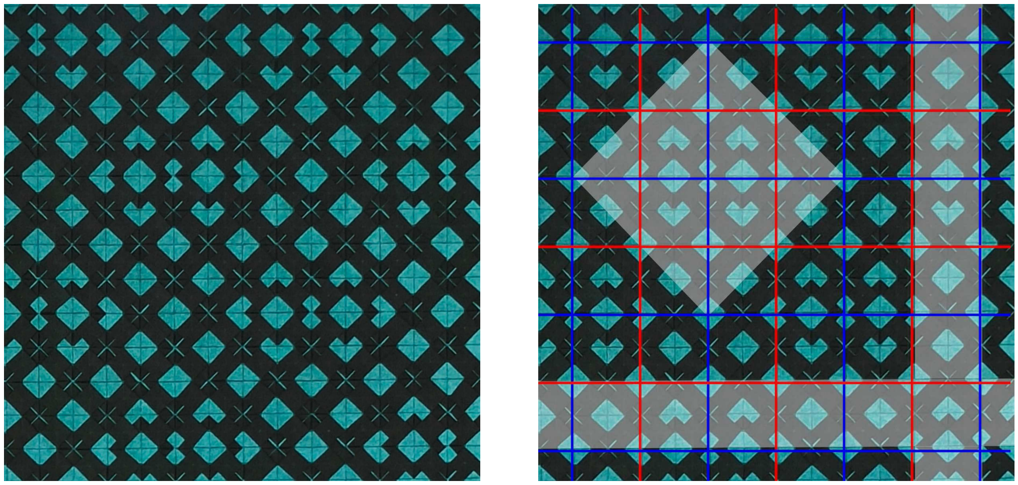


Figure 8: *Origami model of cmm Type 2 (m*1 *and mm)*

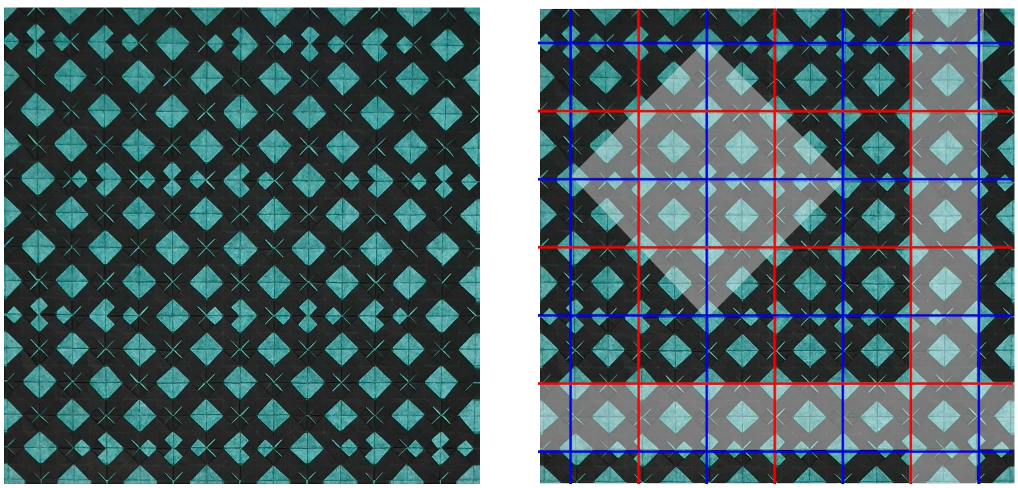


Figure 9: *Origami model of cmm Type 3 (m*1 *and mg)*

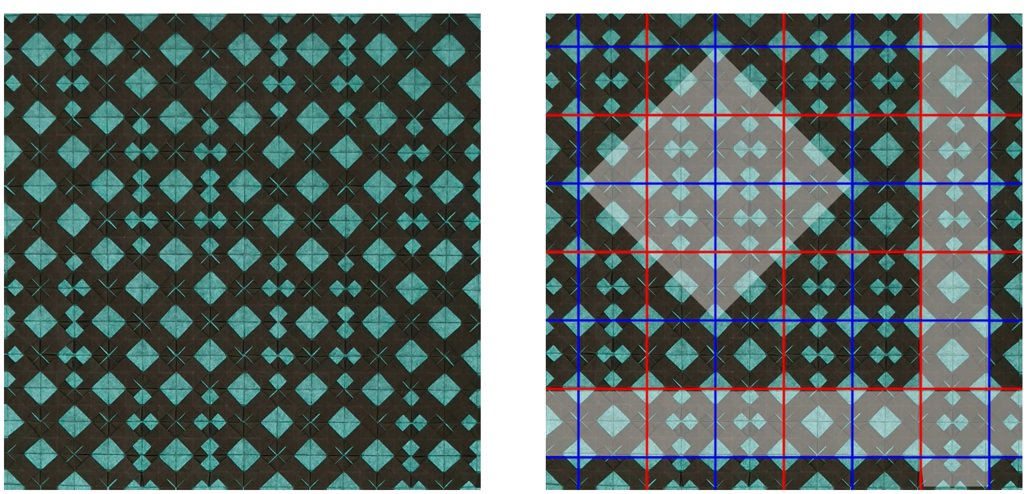


Figure 10: *Origami model of cmm Type 4 (mm and mm)*

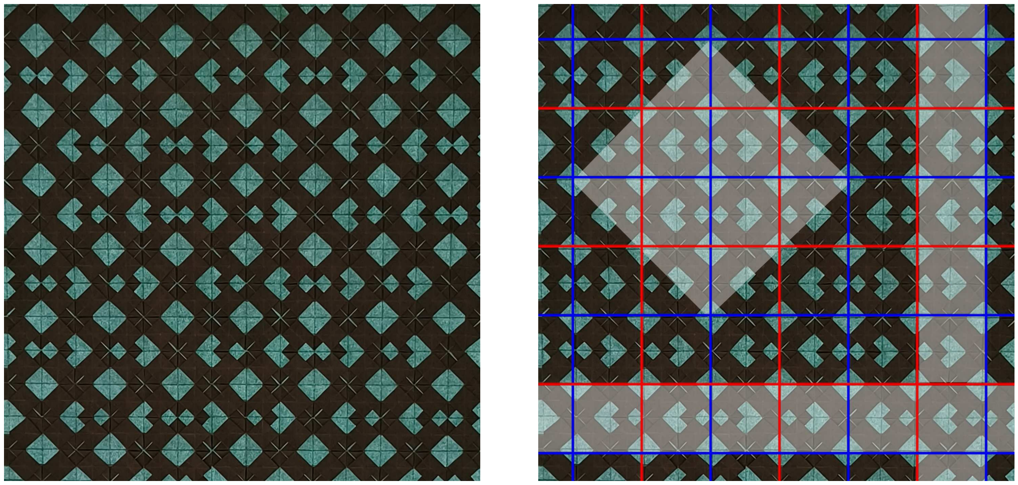


Figure 11: *Origami model of cmm Type 5 (mm and mg)*

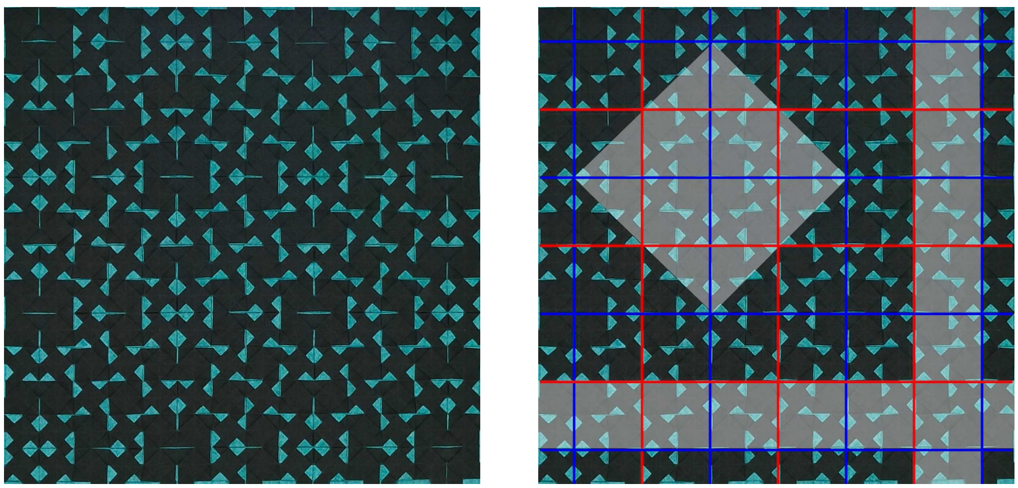


Figure 12: *Origami model of cmm Type 6 (mg and mg)*

Conclusion

This article reports on the first results in a project that aims to classify instrinsic frieze decompositions more generally, and to explore their relationships through adaptable origami designs. Following this study for *cmm*, attention will extend to other wallpaper classes that afford reflection and/or glide reflection symmetries.

Acknowledgement

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References

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[2] R. Fathauer. *Tessellations. Mathematics, Art and Recreation.* CRC Press, 2021.

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