Enhancing Polyhedra, Tilings, and Surfaces with Cone-Studding

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Abstract

For a variety of shapes—plane tilings, polyhedra, and smooth surfaces—we replace a collection of flat kites or rhombi with double cones, a process we call "cone-studding." This technique provides to flat surfaces an intriguing depth and has a variety of applications. For smooth surfaces, where a coordinate system easily gives a grid of quadrilaterals that are almost never planar, we explain the challenges of adapting the technique.

Introduction

The prolific Australian mathematical artist Ghee Beom Kim posted an image that caught my imagination: Faces of a polyhedron had been dissected into kites, and these kites were replaced by pairs of cones—the shape you would get it you revolved the kite about its axis of symmetry. The effect is somewhat similar to my work in Figure 1. I asked him whether he had written about this idea and he said he had not. (Scrolling back to find the original image in his Facebook feed, it appears that his post has been removed, though similar images were posted on November 6 & 7, 2022.) In this article, I investigate the possibilities that arise in various situations when we replace kites, or rhombi in special cases, with pairs of right circular cones that extend on opposite sides of a shared base.

Cone-studding adds a rich texture to ordinary patterns. When rendered in a CAD program such as *Rhino*, my choice, the shapes play nicely with light, creating shadows and throwing reflections in delightful ways. Apply the technique to planar tilings and polyhedra is relatively straightforward, but deep questions arise when we try to apply the technique to curved surfaces, such as the hyperboloid of one sheet in Figure 1b.

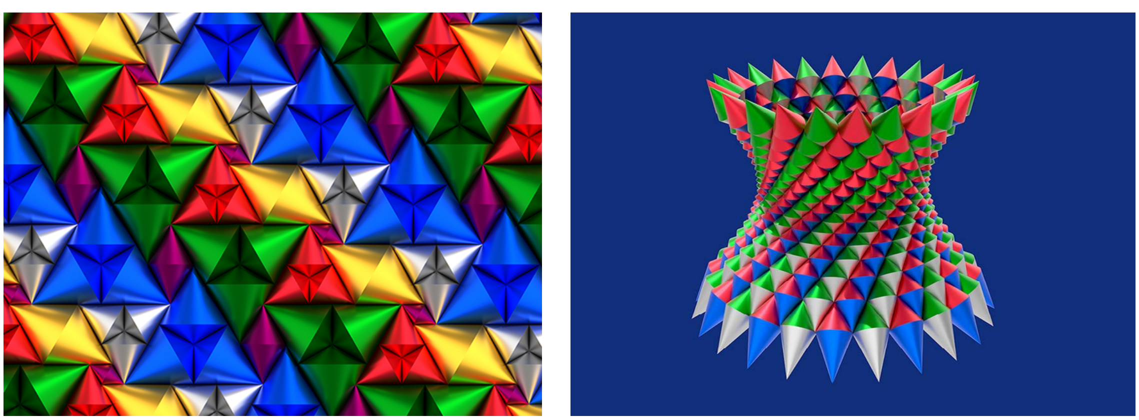

Figure 1: *a) Planar example of replacing kites (or rhombi) with pairs of cones, as process I call cone-studding. b) The technique applied to a hyperboloid of one sheet.*

Since every artistic inquiry deserves some articulated constraint, I will declare that when cones meet, they should share a line segment (or segments) with neighboring cones, so that the complex will appear solid, without unsightly gaps. As we will explain, unless the original pattern is planar, neighboring cones will not be tangent, but will intersect, albeit sometimes quite subtly, as in Figure 1b.

I have tried to include practical information that will assist others in trying their hand at cone-studding. Any CAD program that permits creation of 3D objects with formulas could work. My choice is *Rhino*, the British architectural design program, with *Grasshopper*, a plug-and-play programming environment that is particularly suited for creating lists of 3D objects from formulas. *Grasshopper* offers components that run snippets of Python code, which is useful for the small loop structures needed to create zonohedra. Many of the structures shown in the paper enjoy rotational symmetry; *Grasshopper* makes it easy to construct one portion of these and repeat it by rotation with just one pre-fab component.

Cone-Studding Wallpaper Patterns

Tilings in the plane are a natural place to begin with this activity of replacing kites by pairs of cones. With all the polygons of the pattern in the same plane, there is no concern about cones intersecting their neighbors; as the cones pop up from the plane, they are tangent to neighbors along any line segment shared by two polygons.

It's natural to begin with the best-known tilings. In Figure 2, familiar divisions of the square and hexagonal grids lead to nice examples. Here the kites are all rhombi and a gold band has been set around the equator of the cone pair as decoration. When viewed from the top, these bands appear as line segments, so an oblique view is included to show that their 3D role.

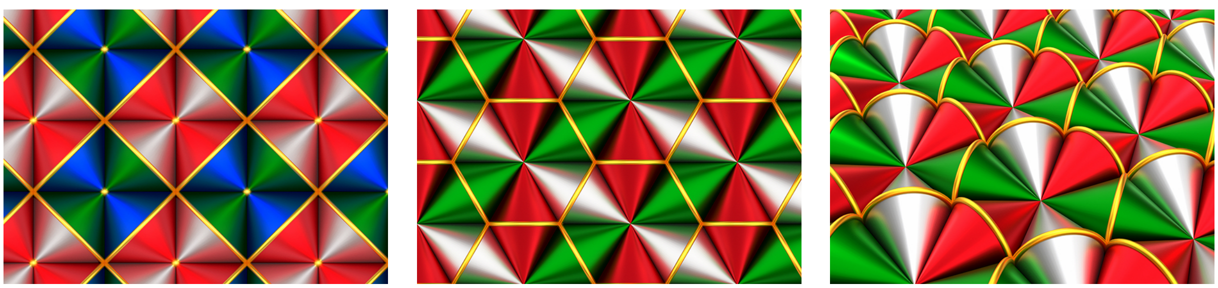

Figure 2: *a) Cone-studding on a square grid. b) Cone-studding on the hex grid. c) Oblique view to show depth.*

As I searched for more interesting tilings, I happened upon some families of tilings that are not edge-to-edge, finding what I believe is a new way to describe the constructions. I explain one in detail and allude to a few other similar outcomes. Readers not interested in the computations may safely skip ahead.

In my effort to construct a repeating pattern with 3-fold rotational symmetry, it was natural to seek a grid of points invariant under rotations of 120° about the points $\omega = -1/2 + \sqrt{3}i/2$ and $\bar{\omega}$. Without some care, generic points will not fall into nice tiling patterns after these rotations. Some experimentation and bungling about led me to consider three related points: a generic point Q, the clockwise rotation of Q through 120° about ω, and the counterclockwise rotation of Q through 120° about $\bar{\omega}$.

Let's name these rotations as

$$\rho_1(z) = \bar{\omega}z - 1 + \omega \text{ and } \rho_2(z) = \omega z - 1 + \bar{\omega}.$$

In general, $Q, \rho_1(Q)$, and $\rho_2(Q)$ will not line up in any special way, and repeatedly translating them in hopes of a nice tiling will just lead to a mess. Fortunately, it turns out that Q being on the unit circle is equivalent to these three points being collinear! In other words,

$$|Q| = 1 \text{ if and only if } Q, \rho_1(Q), \text{ and } \rho_2(Q) \text{ are collinear.}$$

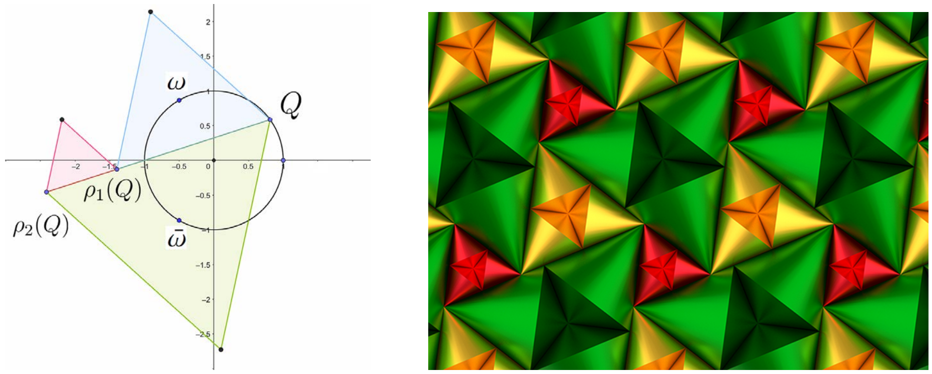

Figure 3: *a) A nice alignment of two rotations of the point Q b) A cone-studding of the resulting tiling.*

This is illustrated in Figure 3a. Moving Q around the unit circle changes the relative sizes of the triangles, producing a continuum of tilings. A representative tiling has been cone-studded for Figure 3b, with the equilateral triangles dissected into three kites each.

This tiling, of course, is not new. It appears in Figure 2.4.2 of Grünbaum and Shephard's *Tiling and Patterns* [5] and Figure 2.25 of Fathauer's newer book, *Tesselations* [4]. Grünbaum and Shephard indicate that one may vary the relative lengths of the sides, as long as the two shorter sides add up to the larger one. However, no simple technique to construct the tiling is mentioned. Investigations similar to the one already described let to several more episodes where a family of tiles can be created by locating a point on a particular circle in the complex plane. Examples appear in Figure 4 and these also appear in Grünbaum and Shephard's Figure 2.4.2 and Fathauer's Figure 2.25. The mathematics behind the two right-hand images is identical; as a parameter slides around a circle in the complex plane, the tiling changes from large hexagons with small triangles to small hexagons with large triangles.

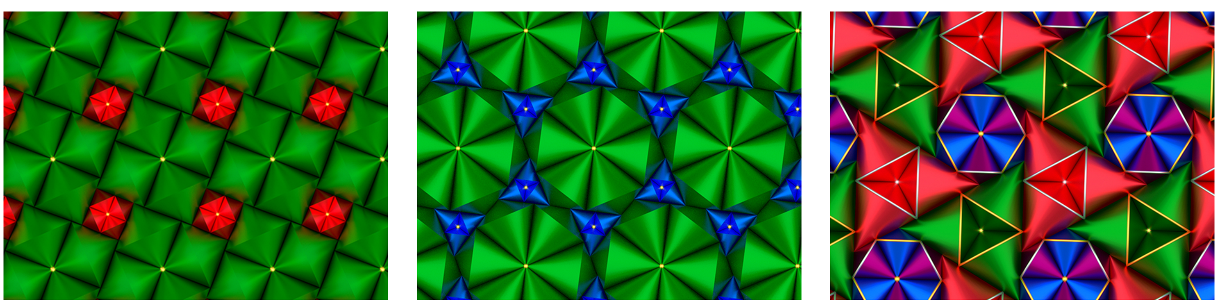

Figure 4: *a) Tiling with squares of two sizes. b)Tiling with hexagons and triangles, dissected into kites. c) The same, but with large triangles and small hexagons.*

Grünbaum and Shepard's Figure 2.4.4 also provided rich ground for cone-studding. Figure 5 shows tilings with 2 and 3 sizes of equilateral triangles. Figure 1a, whose tiling is also in Fathauer's Figure 2.26, starts with a tiling with equilateral triangles of 5 different sizes. When triangles occur in adjacent pairs of the same size, we can treat the pair as a single rhombus, although we could dissect each one into three kites for a different effect. The balance of shapes in Figure 1a seems very pleasing.

For a given tiling, there could be multiple ways to apply the technique of cone-studding. Consider, for instance, the semi-regular tesselation of the plane with squares and equilateral triangles. For a simpler result, each square and each triangle pair could get a double cone, as in Figure 6a. At the other end of the spectrum, each triangle could be dissected into three kites and each square could get a pinwheel of double cones, as in Figure 6b. There are many other possibilities.

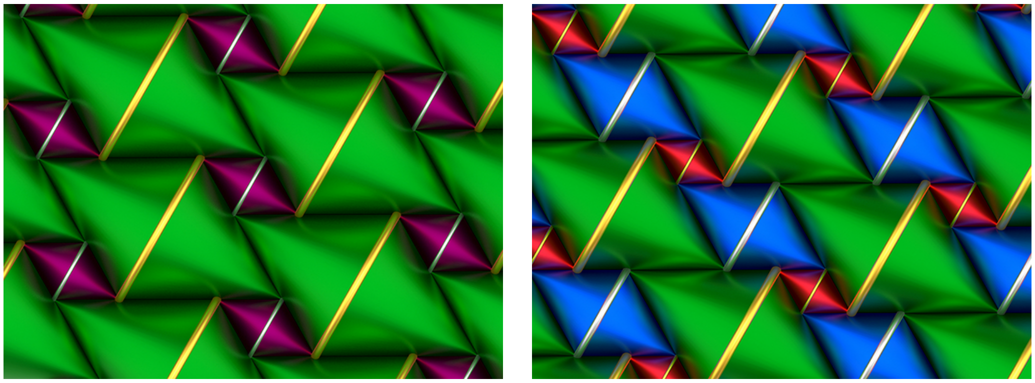

Figure 5: *Cone-studded tilings with 2 and 3 different sizes of equilateral triangles.*

Personally it is a little ironic that I find myself working with tilings after spending so many years explaining why wallpaper patterns should not necessarily be thought of as involving tilings [1]. I still see value in my earlier insistence on creating patterns with organic, wavy lines. But symmetry is symmetry and the delight we seem to take in repeating patterns persists whether there are sharp edges in them or not.

Early in this project, I decided that I should create cone-studded wallpaper patterns for every symmetry type, but am far from having a complete set. Coloring the patterns complicates the goal of a complete sampler, as there are 46 color-reversing types [1]. The pattern in Figure 2a can be seen as having types p4m or p4m/p4m if the blue/green colors are considered the negatives of the red/silver colors. Including three colors makes the list even longer: Ignoring the colors in Figure 2b, we would say it has pattern type p6m, but distinguishing red, green, and silver would suggest a classification as p6m/$_3$p2, where the subscript on the slash is used for 3-color systems [1].

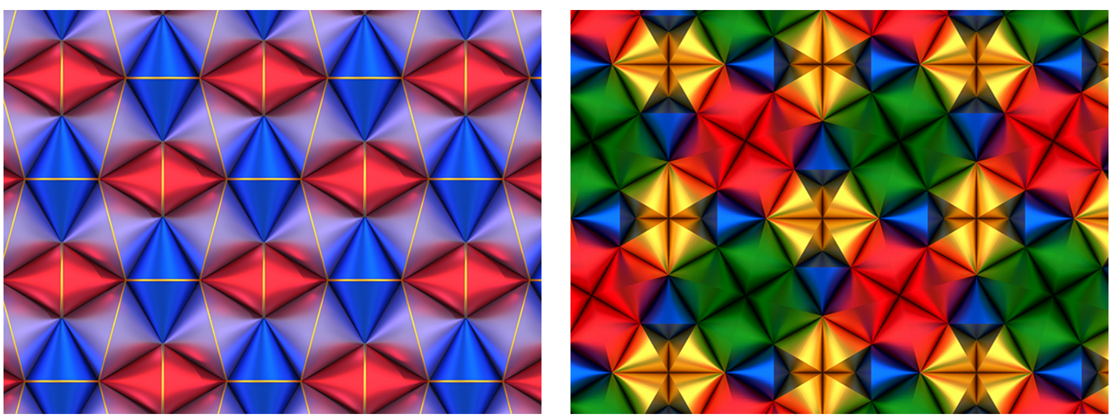

Figure 6: *Two different cone-studding treatments of the same "squares and diamonds" tiling.*

Many of the tilings have high degrees of symmetry: In Figures 4a, 4b, and 4c, we see p4, p6, and a color-reversing pattern p6/p3. Though they are all made with the hex grid as a starting point, Figure 1a and the patterns in Figure 5 all have symmetry type p2, although we need to ignore color in Figure 1a for that classification; as a color-reversing pattern it can exemplify p2/p1, counting the purple color as the negative of itself. Considering only the tiling in Figure 6, we would identify the symmetry group as p4g. However, the cones in Figure 6a break the rotational symmetry of the squares, resulting in a pattern invariant under the group cmm. The color group of the pattern in Figure 6b is p4g, treating red and green as opposite colors, with gold and blue as neutral colors. The symmetry group is p4, so the type of that pattern is p4g/p4.

Clearly, there remain many pattern types yet to produce. Another expansion could be to tilt the cones up and down from the plane, to create patterns invariant under the 80 layer groups [2], which are closely connected to, but different from, the color group pairs.

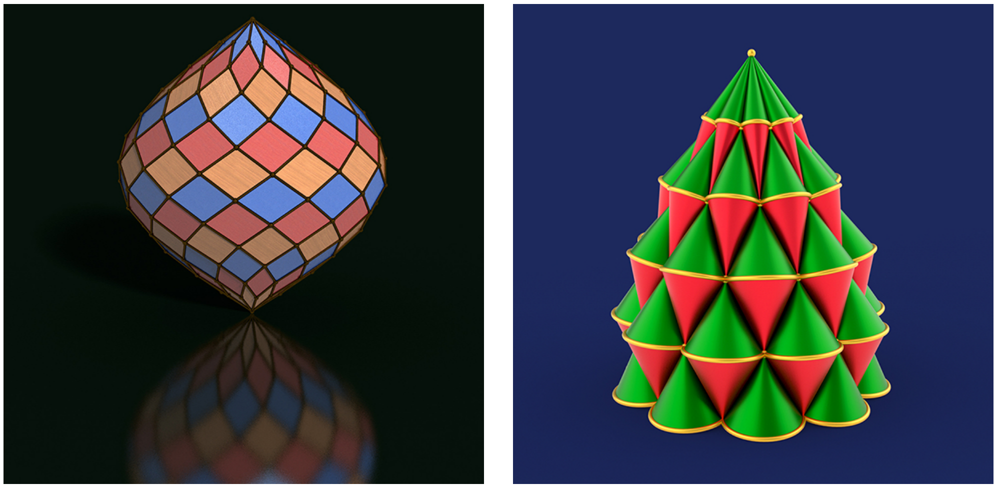

Figure 7: *a) A typical polar zonohedron, with 8-fold rotational symmetry and a vertex angle of* 45°. *b) A truncated zonohedron with 9-fold rotational symmetry with Christmas colors.*

Cone-Studding Polar Zonohedra

George Hart's paper "The Joy of Polar Zonohedra" was a highlight of the 2021 Bridges conference [6]. I spent that autumn creating zonohedra in Rhino with Grasshopper. Recall that these rotationally symmetric polyhedra start with a symmetric "umbrella" of n unit vectors based at the origin and all inclined at a common angle α with the vertical; the polyhedron is built by repeated addition of pairs of adjacent vectors, as in Figure 7a. Since every zonohedron is composed of rhombi, they were a natural choice for my first cone-studding experiments. Figure 7b shows an example that came just in time for Christmas.

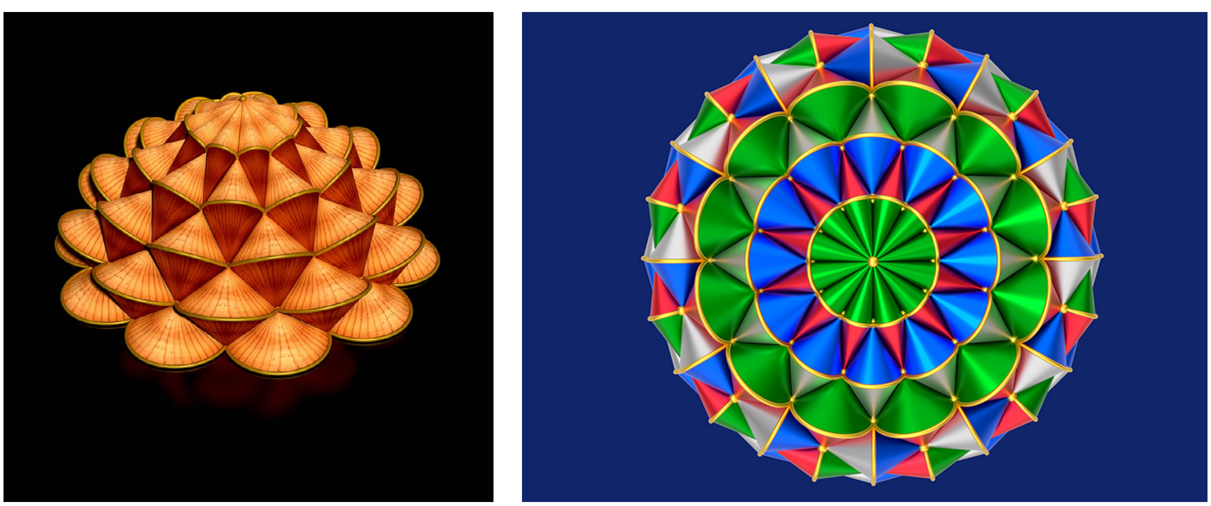

Figure 8: *Cone-studded zonohedra: a) decorated with wood grain. b) with the cone vertices placed along the longest axis of the rhombi.*

Possibilities for coloring cone-studded zonohedra are endless. Figure 8a shows the top half of a zonohedron with 10-fold symmetry color with dark and light wood grains, along with golden bands and balls at the tips of cones. In Figure 8b, cones are inserted with the vertices placed on the long axes of the rhombi. This keeps the cones from sticking out too far from the original polyhedral surface; without this adaptation, the cones around the waist of a zonohedra of high rotational symmetry can end up looking a little like coins instead of cones.

A Closer Look at Neighboring Cones

Try this thought experiment, perhaps before looking at Figure 9: Imagine two cones that share an edge segment but not a vertex; this is the situation for adjacent cones on the zonohedra shown. If the axes of the cones lay in the same plane–say, the x-y plane—the cones would be tangent. Now imagine tilting one of the cones upward out of that plane, using the tangent segment as a hinge. As soon as the cones are no longer tangent, they must intersect in a nonzero volume, with the volume increasing as the cone is tilted more. The situation is illustrated in Figure 9.

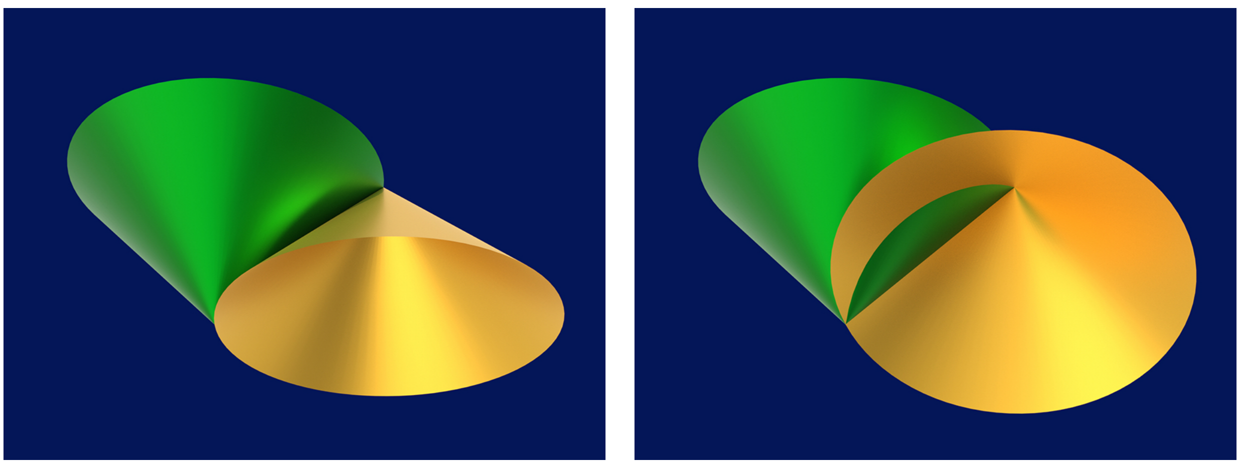

Figure 9: *a) Cones with coplanar axes. b) Cones that share a segment with axes in different planes.*

In my first experiments with zonohedra, like those shown in Figures 7 and 7, I naively imagined that the cones I was creating were all just tangent to one another. Of course they are not, but the overlapping volumes are hidden on the inside of the complex. I considered whether I could somehow lift the cones away from the center of the zonohedra to achieve tangency, but this seems impossible without moving the vertices in a way that would leave gaps. This could be a fruitful direction for future work.

Cone-Studding Smooth Surfaces

The most interesting direction for future work seems to involve applying the technique of cone-studding to general surfaces, as a visually stimulating way to thicken surfaces. Several questions arise, but I have only partially formulated them. In this section, I report some successes.

Mercator coordinates on the sphere gave nice results, as seen in Figure 10a. The parametrization is

$$\vec{x}(u, v) = (\cos(u)\,\mathrm{sech}(v), \sin(u)\,\mathrm{sech}(v), \tanh(v)),$$

where $0 < u < 2\pi$ and $0 < v < \infty$. The advantage of these coordinates is that the velocity vectors of the u-curves and v-curves are equal at every point, meaning that the coordinate system is *conformally equivalent* to the Euclidean system, with equal stretches of length in all directions.

A rectangular grid in the u-v plane can be adapted to a rhombic grid. The coordinate map allows us to paste those rhombi up onto the sphere, where they form quadrilaterals that are not quite planar. With a small enough grid, the quadrilaterals are close enough to planar rhombi to give nice results with the cone-studding code I used for zonohedra. For Figure 10a, I built a single stack of rhombi circling up around the sphere and repeated it multiply by rotation. A *Grasshopper* component makes this easy work.

In any application of cone-studding, we will have vertices A, B, C, and D of a quadrilateral where we wish to house a double cone. Let's say that the two vertices of the cone will be A and C, meaning that points B and D should lie on the circular base of the cone. If ABCD is a kite (including a square or rhombus), locating the cone pair is easy: Create a circle in the plane with normal vector $\mathrm{A} - \mathrm{C}$, center $(\mathrm{B} + \mathrm{D})/2$, and radius $|\mathrm{B} - \mathrm{D}|/2$. The circle and the vertices on either side give a mathematical specification of the two cones, but how are they constructed in software? How would they be made in the real world? After creating the circle I used a *Grasshopper* component to subdivide it into a dozen or so equally spaced points, created lines from the vertex to those points on the circle, and then created what is called a *loft surface* from that collection of lines. This is an example of how *Grasshopper* allows us to move quickly from a mathematical specification of an object to a nicely-meshed 3D one.

Things are a little more difficulty when quadrilateral ABCD is not planar. In order to obey the rule about leaving no gaps between cones, we could connect cone vertices to any circle containing B and D, but I made a symmetric choice for the center and radius to remain $(\mathrm{B} + \mathrm{D})/2$ and $|\mathrm{B} - \mathrm{D}|/2$. What about the normal vector to the plane? I made a somewhat symmetric choice in

$$N = ((\mathrm{A} - \mathrm{C}) \times (\mathrm{B} - \mathrm{D})) \times (\mathrm{B} - \mathrm{D}).$$

This produced the results shown, where the quadrilateral is close to being a planar kite.

Figure 1b shows this approach on the hyperboloid of one sheet. I created a stack of cones along one ruling and rotated it around. In all examples of surfaces, a smaller rhombic grid will produce a shape that more closely adheres to the given surface.

For a non-rotational example, I used the hyperbolic paraboloid, or saddle surface. This time I used a complex coordinate in the x-y plane; the ruling patch is

$$\vec{x}(u, v) = (e^{iu} + ive^{iu}, v).$$

Figure 10b shows an example with cones inserted in a fairly fine grid of quadrilaterals on the surface.

Theoretically, one can find conformally Euclidean (or *isothermal*) coordinates on any smooth surface, but it may be difficult in practice. Even without such nice coordinates, one can sprinkle cones anywhere. Using the technique described above for inserting cones in non-planar quadrilaterals, cones can be made to share a segment, which I suggest as a niceness condition for cone-studding.

One interesting question is: Can cones be inserted in a grid on a surface in such a way that the cones are tangent? I do not know. But the routines that I have described might be interesting in a different way with cones that are not tangent: Can the Gaussian curvature of the surface be estimated via the overlap of adjacent cones on either side of the normal? I leave this for future work.

Summary and Conclusions

The technique of cone-studding can be applied to many more tilings, polyhedra, and smooth surfaces than I have had the chance to investigate. The mathematical substance of the idea is probably less than the artistic potential. I hope that others will build on this work and develop their own examples.

A sampler of cone-studding images for each kind of wallpaper symmetry, or for all the tilings of a certain type, could make a nice project. Since the double cones resemble beads, and since many of the tilings use cones of only a few different sizes, there is potential for realizing those shapes with beads.

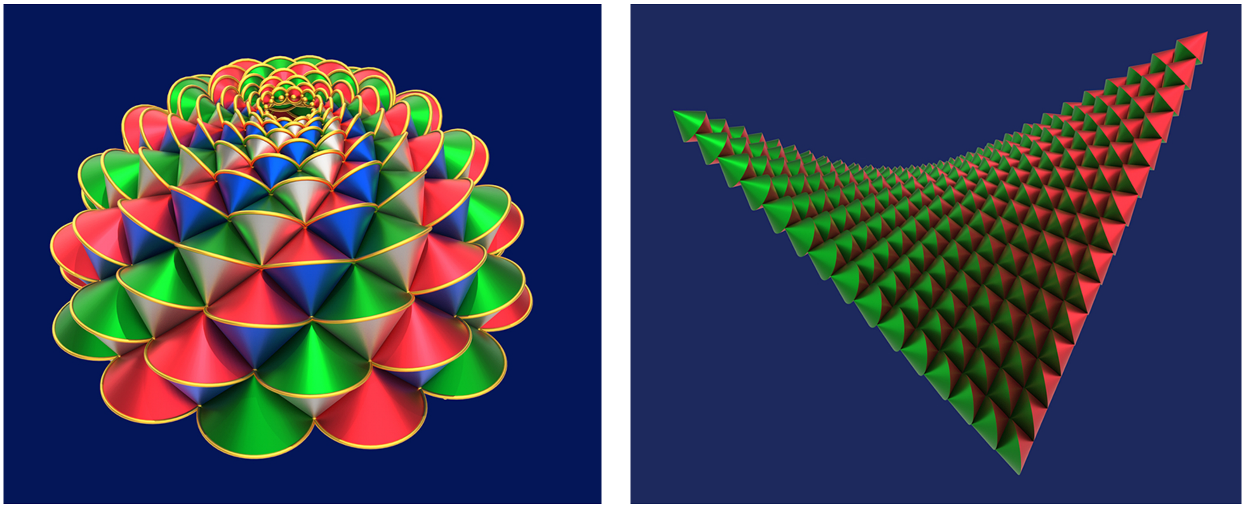

Figure 10: *a) A cone-studded sphere, where the cones lie along loxodromes. b) Cones set on a hyperbolic paraboloid.(In both cases, neighboring cones may appear tangent, but they are not.)*

After making so many virtual images of these assemblies of cones, my next step should be to realize some of them as 3D prints, whether in metal or ceramic. I have started joint work with ceramic artist Timea Tihanyi, but so far none of the shapes from this paper have crossed the boundary from *Grasshopper* into the real world.

Acknowledgements

Thanks to Ghee Beom Kim for the original inspiration for cone-studding. And thanks to Gwen Fisher for helpful advice.

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