

Artsy Pseudo-Hamiltonian Tours

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Abstract

A variation on Hamiltonian cycles in graphs are found in—or can be applied to—designs in a variety of art forms such as dance and educational kinesthetic activities, polyrhythms, lace shirtwaist buttons, paths of billiard balls in polygons, Celtic knots, and string figures. We examine a variety of what may be idiomatically called pseudo-Hamiltonian tours, but which we are describing as $\{n/(a_1, a_2, a_3, \dots, a_m)$ designs. We suggest some applications, with special attention to their use in embodied mathematics movement explorations. These designs can help provide crossover points for visual and movement artists and mathematicians to make connections between their fields.

Introduction

What might a game with playground balls, partner dances, polyrhythms, billiard ball paths, Celtic knots, string figures, and lace shirtwaist buttons have in common? These all may incorporate designs that pop up in a variety of arts and arts oriented activities. Appearances of the designs in dance involve the location of dance partners in a circular formation, as partners travel around the circle. Or in circular displays of polyrhythms. We may see a correspondence between certain billiard ball patterns, Celtic knots, and geometric string figures. The designs display connections to recently explored variations on Hamiltonian tours in graphs. We will examine several of these designs and suggest further explorations. They may aid artists and mathematicians in developing sometimes surprising connections between their fields.

Ball Passing Problems

In November 2022, the first author was resident for a week at the University of Wisconsin–Madison, in an NSF funded project collaborating with the second author, working together on exploring kinesthetic activities that involve mathematical problem solving. Our goal was to explore aspects of rhythmic movement patterns and their relation to mathematical thinking and learning. The collaboration initially consisted of weekly one-hour Zoom meetings, and the week in Madison included daily meetings Monday through Thursday plus a day and a half workshop for approximately ten participants on Friday, Oct. 28 and Saturday, Oct. 29, 2022. We also met several times with Michelle Ramos, dance faculty member in the UW Dance Department and planned a visit by workshop participants to her Friday ballet class.

Our discussions and workshop activities included explorations of two types of rhythmic composition: the decomposition of rhythmic structures according to the lengths of component sections of a rhythm, and the creation of polyrhythmic patterns that overlay two periodic patterns. In the Saturday workshop we engaged in a ball passing game that kinesthetically employs structures found in polyrhythms. The ball-passing activity led to interesting mathematical and movement explorations and problem-solving, and included examinations of symbolic and algebraic forms for representing the activity. That activity had participants standing in a circle periodically passing and bouncing rubber playground balls around the circle, while keeping track of who were the starting and ending participants as the passes proceeded.

The ball passing activity shares mathematical connections with certain mathematical properties of n -cycles called star polygons that also appear in polyrhythms [10,ch.2]. Suppose ten participants stand equally spaced in a circular formation as shown in Figure 1(a), with participants labeled 0 through 9, the so-called *least residues* mod 10, around the circle. In this modular arithmetic labeling scheme participant 0 could equally be called participant 10. Participant 0 begins passing a ball to the second person clockwise, number 2, and participants continue passing to the second person clockwise. Note that the ball returns to 0 after five

passes. At the same time person 5, for example, might similarly pass a second ball 2 participants clockwise. Two interlocking pentagonal paths of the balls are created. Together these are called the $\{10/2\}$ (compound) star polygon, using what is known as Schläfli notation. Every vertex has degree 2 in such a configuration. Figure 1(b) shows the $\{10/3\}$ star polygon, in which the ball is passed 3 participants clockwise on each pass. In the $\{10/3\}$ sequence that one ball reaches each participant, while the $\{10/2\}$ creates disjoint pentagonal paths, one to the even numbered participants, the other to the odd numbers. These patterns are well known and are described by the following proposition utilizing the greatest common divisor (gcd) of the number of participants n and the number of participants forward for each pass, k [2, pp. 36–38]. We use directed edges shown by arrows in our discussion, though we often leave out the arrowheads if the sequence is clear.

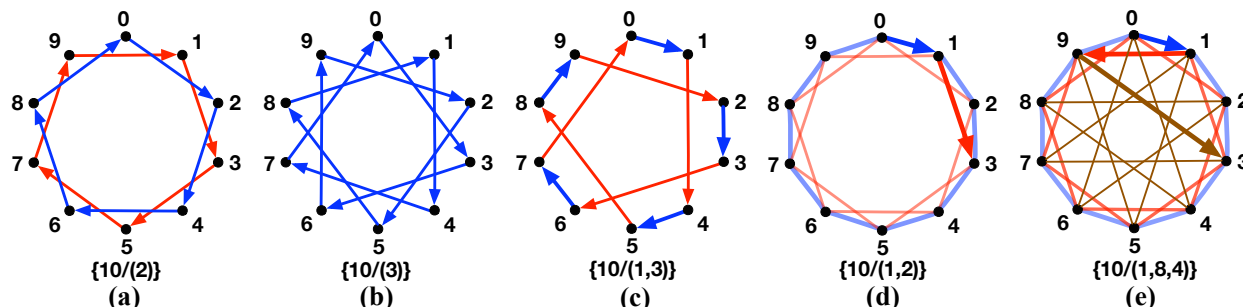


Figure 1: (a) and (b) star polygons. (c), (d), and (e) use an extension of the Schläfli notation in which edges alternately jump 1 then 3 places; 1 then 2 places; and 1, 8, and 4 places, clockwise respectively. We use thick blue, medium thickness red, and thin brown edges in (c), (d), and (e) for better visibility.

Proposition 1. The $\{n/k\}$ star polygon using the vertices of the regular n -gon is composed of $\gcd(n,k)$ copies of disjoint regular $\frac{n}{\gcd(n,k)}$ -gons. Each such polygon circles the n -gon’s center $\frac{k}{\gcd(n,k)} = \frac{\text{lcm}(n,k)}{n}$ times, where $\text{lcm}(n,k)$ is the least common multiple of n and k , and $\gcd(n,k)$ is their greatest common divisor.

We will discuss a generalization of star polygons using an extension of Schläfli notation that we are calling $\{n/(a_1, a_2, a_3, \dots, a_m)\}$ designs. Care needs to be taken in the definitions of these concepts. We identify and label the vertices of C_n with the cyclic group Z/nZ , the integers modulo n , where those vertices are equally spaced clockwise from 0 to $n - 1$ around a circle. We define a directed edge (x, y) between two vertices x and y of C_n as the edge from vertex x to vertex y . We say the weight of the edge is the element of the set of residues mod n , $\{0, 1, 2, \dots, n - 1\}$, that is congruent to $y - x \pmod{n}$. Intuitively, this is the number of vertices from $x + 1$ to y inclusive around C_n in the clockwise direction. We will not include a loop or single edge of weight zero from a vertex x to itself in our discussion, though the sum of the weights of a circuit with more than one edge derived from C_n may be congruent to zero, mod n . For calculations involving the vertices we will freely use modular arithmetic and if a vertex label is k we will simply call that vertex k , though it may be convenient at times to use another integer congruent to k , mod n .

Definition. Letting $A_m = (a_1, a_2, a_3, \dots, a_m)$ we define the design $\{n/A_m\}$ to be a sequence of directed edges between vertices of C_n of weights in the order $a_1, a_2, a_3, \dots, a_m$ that starts at a given vertex, often 0, and repeats the sequence A_m until the edge with weight a_m first terminates at the starting vertex.

Note that the vertices of $\{n/A_m\}$ are those of C_n though $\{n/A_m\}$ may or may not include directed edges belonging to the cycle C_n . The collection of these directed edges derived from C_n form a multigraph in which two vertices may be connected by more than one edge. If the design $\{n/A_m\}$ is called on to repeat an edge from vertex x to y then that requires a new edge since $\{n/A_m\}$ forms what is known as a *circuit* in graph theory, in that it is a closed walk that may repeat vertices but not edges. We may also freely substitute an edge weight a_i by another integer congruent to a_i , mod n . It may be more convenient, for example, to refer to the edge in C_{10} from 0 to 8 as the edge of weight -2 , since $-2 \equiv 8 \pmod{10}$ and it may be easier to

visualize an edge going two steps counterclockwise rather than one going eight steps clockwise. Similarly, the directed edge from 8 to 0 in C_{10} will have the weight 2.

In Figures 1 (c), (d), and (e) the design may represent passes of a ball along edges of alternating weights clockwise around C_{10} . For example, $\{10/(1,3)\}$ indicates that the sequence of passes or edges clockwise around C_{10} starting at 0 is 1,3,1,3,1,3,..., This corresponds to the new idea the paper [8] introduces as a (1,3)-step Hamiltonian cycle or tour of the 10-cycle C_{10} . Each vertex in C_{10} is visited once by this cycle. Yet we do not include in this design every pass or edge of weight 1 or 3, for example the weight 3 edge from vertex 2 to 5. Here the thick blue weight 1 edges make up a “perfect matching” in which each vertex has degree 1. The thinner red weight 3 edges are also a perfect matching, and in fact this partition into two perfect matchings is characteristic of every $\{n/(a,b)\}$ cycle or design that visits each vertex exactly once in any graph with an even number of vertices.

Figure 1(d) is a $\{10/(1,2)\}$ design. Its seventh edge ends at vertex 0 and has weight 1. However, for its sequence of edges to first end at 0 with an edge of weight 2 so it then seamlessly continues with an edge of weight 1, $\{10/(1,2)\}$ visits each vertex twice; we denote this $\{10/(1,2)\}$ design as an example of a (1,2)-step double- or 2-Hamiltonian tour of the vertices of C_{10} . Figure 1(e) is an example of a $\{10/(1,8,4)\}$ design or a (1,8,4)-step triple- or 3-Hamiltonian tour of C_{10} which visits each vertex three times.

Definition. An $\{n/A_m\}$ -step k -Hamiltonian tour is a sequence of edges between the vertices of C_n of weights $a_1, a_2, a_3, \dots, a_m$ that repeats that sequence until the edge with weight a_m first terminates at the starting vertex and such that this circuit visits each of the vertices of C_n k times.

An $\{n/A_m\}$ -step k -Hamiltonian tour may also be called a *pseudo-Hamiltonian tour* [1], though we are also using the adjective *pseudo* idiomatically to indicate $\{n/A_m\}$ designs that generate configurations of general artistic interest.

The paper [8] introduces and explores these concepts in all connected graphs G , not only cycles as in this paper; it assigns the edges (x,y) and (y,x) the same weights, namely the graphical distance between x and y in G , which is the number of edges in the shortest path in G between them. Thus in that framework both edges $(0,8)$ and $(8,0)$ in C_{10} have weights 2. A result is that in [8] there are no edges at all with weights 8 between vertices of C_{10} since the maximum graphical distance between the vertices of C_{10} is 5.

In our Madison workshop we complicated the passing rules to create problem solving that combined kinesthetic whole body engagement with mathematical concepts. The goal was to kinesthetically embody the simultaneous playing of two periodic rhythmic patterns with distinct periods that form a polyrhythm. For example, in one exercise, shown in Figure 2(a), Erica began a sequence of passes of the red ball three participants clockwise while Andrea began a sequence passing the blue ball two participants clockwise, both with the same tempo so the passes of the two balls occurred together. This duplicates a polyrhythm in which 2 beat and 3 beat rhythms are played against an overall 5 beat pattern, with distinct starting down beats for the 2s and 3s. When might the balls, or down beats, arrive at the same person at the same time? Participants used the following four methods to successfully answer this question, but also to consider other mathematical or movement ideas that might arise:

1. Simply trying it and seeing what happened and what skills were necessary (for example, if the balls seemed likely to hit in the air, we bounced one and threw the other over it!)
2. Tabulating a list of people the balls reached on each pass, see Table 1.
3. Drawing diagrams like Figure 2(a); the small diagrams help us predict when the balls might collide!
4. Using modular arithmetic to solve the problem symbolically: Letting x represent the number of passes needed for the two balls to reach the same person, and assuming the positive direction around the circle or pentagon is clockwise, note that balls start at vertices 0 and 1 and that $3x \equiv -2x \pmod{5}$. Here we use the multiplicative inverse in which $4^{-1} \equiv 4$ since $4 \cdot 4 = 16 \equiv 1 \pmod{5}$:

$$\begin{aligned}
 0 + 2x &\equiv 1 + 3x \pmod{5} \\
 4x &\equiv 1 \pmod{5} && \text{[Subtract } 3x \text{ from both sides]} \\
 4^{-1} \cdot 4x &\equiv x \equiv 4^{-1} \cdot 1 \equiv 4 \pmod{5} && \text{[Multiply both sides by } 4^{-1}]
 \end{aligned}$$

This problem has a standard mathematical solution involving linear congruences in modular arithmetic. If the blue ball starts at person a and is passed clockwise b participants on each pass, and the red ball starts at person c and is passed clockwise d participants on each pass, in an n person circle, then the congruence equation $a + bx \equiv c + dx \pmod{n}$ has solutions given by the following proposition. See for example [3, ch. 5.2].

Proposition 2. The linear congruence $a + bx \equiv c + dx \pmod{n}$ has solutions, mod n , if and only if $e = \gcd(b - d, n)$ is a divisor of $c - a$. We then solve:

$$\begin{aligned}
 (b - d)x &\equiv (c - a) \pmod{n} \text{ by finding} \\
 f &\equiv \left(\frac{c-a}{e}\right) \left(\frac{b-d}{e}\right)^{-1} \pmod{\frac{n}{e}} \\
 \text{with solutions } x &\equiv f + \left(\frac{n}{e}\right)k \pmod{n}, \text{ for } k = 0, 1, 2, \dots, e - 1
 \end{aligned}$$

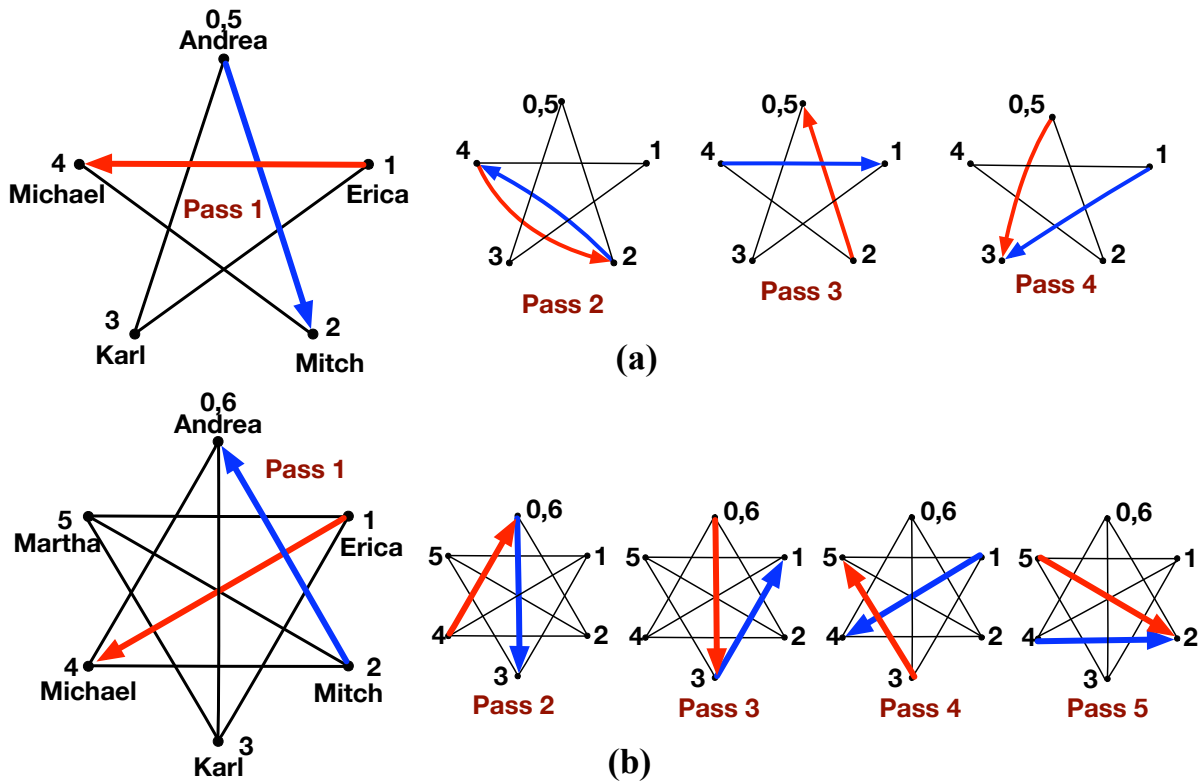


Figure 2: (a) $\{5/2\}$ blue edge and $\{5/3\}$ red edge designs meet at vertex 3. (b) $\{6/(3,2)\}$ red edge and $\{6/(4,3)\}$ blue edge designs meet at vertex 2. (Names are those of the actual participants.)

Table 1: $\{5/2\}$ and $\{5/3\}$ passing patterns.

Pass number:	0	1	2	3	4
Person holding blue ball at end of pass:	Andrea	Mitch	Michael	Erica	Karl
Vertex of blue ball at end of pass	0	2	4	1	3
Person holding red ball at end of pass:	Erica	Michael	Mitch	Andrea	Karl
Vertex of red ball at end of pass	1	4	2	0	3

For example, in Figure 2(a), $a = 0, b = 2, c = 1, d = -2, n = 5, e = 1, f \equiv 4 \pmod{5}$. An example that might be visualized using a decagon (not shown) is $a = 1, b = -2, c = 3, d = 4, n = 10, e = 2, f \equiv 3 \pmod{5}$, and $x \equiv 3 \text{ or } 8 \pmod{10}$. (When we use the $=$ symbol here instead of \equiv we are usually restricting discussion to plus or minus the “residues,” mod n , which are the non-negative integers from 0 to $n-1$.)

We can easily characterize $\{n/(a,b)\}$ designs that create (a,b) -step Hamiltonian tours that visit each vertex once in C_n . For example, pick two odd numbers, say 3 and 5. Then pick a third number, say 10, such that the $\gcd(10,3+5) = 2$. Then $\{10/(3,5)\}$ will be a $(3,5)$ -step Hamiltonian tour in C_{10} . See Figure 3(c) which is a plan for a $10/(3,5)$ ten-finger string figure based on a billiard ball artwork modeling a periodically bouncing ball in a regular pentagon where angle of incidence equals angle of reflection created by Diana Davis [5]. This $\{n/(a,b)\}$ design pattern generalizes to include $\{n/(a_1, a_2, a_3, \dots, a_m)\}$ designs:

Theorem 1. Label the vertices of C_n using the residues, mod n , $\{0, 1, 2, \dots, n-1\}$. Then $\{n/(a_1, a_2, a_3, \dots, a_m)\}$ is an $(a_1, a_2, a_3, \dots, a_m)$ -step Hamiltonian tour of C_n if and only if the following conditions hold:
 1. $\gcd(n, a_1 + a_2 + a_3 + \dots + a_m) = m$.
 2. The m sums $a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+a_3 + \dots + a_m$ are distinct, mod m .

For a proof with examples and further results, see the Supplement to this paper in the Bridges Archives.

In the Madison workshop we investigated the following problem connected to (a,b) -step Hamiltonian cycles. Six participants, shown in Figure 2(b), again passed playground balls in unison, but each ball was passed using an alternating sequence of participant skips. The red ball, shown by the red arrows, was passed starting at vertex 1, by Erica, using the pass sequence 3, 2, 3, 2, The blue ball was passed starting at vertex 2, by Mitch, using the sequence 4, 3, 4, 3, Note that we might also interpret a pass of +4 as equivalent to -2 , since $-2 \equiv 4 \pmod{6}$. We asked when, if at all, do the balls return to the starting person? When, if at all, do they arrive simultaneously at the same person? What mathematical and movement ideas might we observe and employ? Again, we first simply tried performing the pattern, becoming more adept at the movement task as we worked. Table 2 shows the locations of the balls after each pass. Algebraic linear congruence methods also allow us to calculate when the two patterns send the balls to the same person at the same time or when they begin a repetition of the pattern — see the explanation in the Supplement to this paper. Both balls arrive at the same person at the end of $6k + 5$ passes for any non-negative integer k .

Table 2: $\{6/(3,2)\}$ and $\{6/(4,3)\}$ passing patterns.

Total number of passes	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Vertex of red ball	1	4	0	3	5	2	4	1	3	0	2	5	1	4	0	3	5	2	4
Vertex of blue ball	2	0	3	1	4	2	5	3	0	4	1	5	2	0	3	1	4	2	5

Table 3 shows the results of experiments using two ball passing patterns chosen from $\{6/(2,3)\}$, $\{6/(3,2)\}$, $\{6/(4,3)\}$, and $\{6/(3,4)\}$, one of which is started at position 0 and the other at position 1 in a 6-person circle. An ∞ symbol indicates the balls never meet at the same person. The notation $k \pmod{n}$ indicates the balls reach the same person at the end of pass $k \pmod{n}$.

Table 3: Several combined alternating patterns for 6-person circle.

Start at 0 → Start at 1 ↓	$\{6/(2,3)\}$	$\{6/(3,2)\}$	$\{6/(4,3)\}$	$\{6/(3,4)\}$
$\{6/(2,3)\}$	∞	$1 \pmod{2}$	∞	$1 \pmod{6}$
$\{6/(3,2)\}$	∞	∞	∞	∞
$\{6/(4,3)\}$	∞	$5 \pmod{6}$	∞	∞
$\{6/(3,4)\}$	$5 \pmod{6}$	∞	$1 \pmod{2}$	∞

What movement skills and maneuvers are necessary? What connections do participants make between the embodied activity and the mathematical concepts — or rhythmic or other artistic patterns? What patterns

are discernable for distinct numbers of participants, different starting positions, and varied $\{n/(a,b)\}$ patterns? These questions could make engaging embodied math investigations in a variety of courses.

Polyrhythms, Circular Dance Patterns, Celtic Knots, and String Figures

Graphs with the undirected edges of Figures 1(a), (b), (d), and (e) are known as *circulant graphs* in which vertices are labelled $0, 1, 2, \dots, n-1$, such that if vertices v and $v + d \pmod{n}$ are joined by an edge then so are all vertices x and $x + d \pmod{n}$. These circulant graphs are often used to display polyrhythms. Figure 1(a) may be seen as a two-beat rhythmic pattern played over a ten-beat rhythm, the two-beat pattern forming two disjoint pentagons, one starting on an even vertex (even vertex beats might be considered the down beats of the ten-beat pattern), the other on an odd vertex. Figure 1(b) might show a three-beat rhythm played against a ten-beat rhythm. Figure 1(d) shows a three-beat rhythm, each bar of which is composed of 1 beat plus 2 more beats; it circles the diagram $\frac{3}{\gcd(3,10)} = 3$ times before beginning again at the starting point zero.

Figure 1(e) exemplifies a difference between the use of these figures to represent kinesthetic versus rhythmic patterns. Since $8 \equiv -2 \pmod{10}$, a ball passing or dance partner switching routine might have the mover (the ball or the dancer) move clockwise $1 - 2 + 4 \equiv 3$ places on each $(1,-2,4)$ phrase and thus circle the figure 3 times before beginning to repeat the pattern. As a rhythmic phrase, each $(1,8,4)$ phrase represents $1 + 8 + 4 = 13$ beats, so the figure illustrates a polyrhythm combining 13 beat and 10 beat simple rhythms. Thus the 13 beat phrase begins its polyrhythmic pattern again with respect to the 10-beat phrase after “circling” the 10-beat 10-cycle $\frac{13}{\gcd(13,10)} = 13$ times.

These designs are also very simple versions of possible juggling patterns. However, each juggler would most likely pass juggling implements on each beat, and jugglers do not typically limit themselves to circular formations. The designs may represent paths dance partners take in circular dance formations, in which case every position may also represent a partner about to move to a new position on each musical phrase. For discussion of connections between juggling patterns and partner dance switching often using $\{n/k\}$ polygon patterns, see [13], which mentions, for example, the dance Bingo that uses an $\{n/4\}$ pattern. In [9] Christine von Renesse details classroom activities exploring partner paths of the form $\{n/k\}$ and mentions having also explored the $\{n/(1,2)\}$ pattern with math and movement classes in the Salsa Rueda dance form. She also suggests investigating $\{n/(a,b,c)\}$ dance patterns. Folk dance mixers are sometimes simple, having partners move one person clockwise or counterclockwise in the circle to find new partners, but often involve randomness in how dancers move to the next partner. An interesting further exploration would be to catalogue those mixers that use repeating but complex patterns.

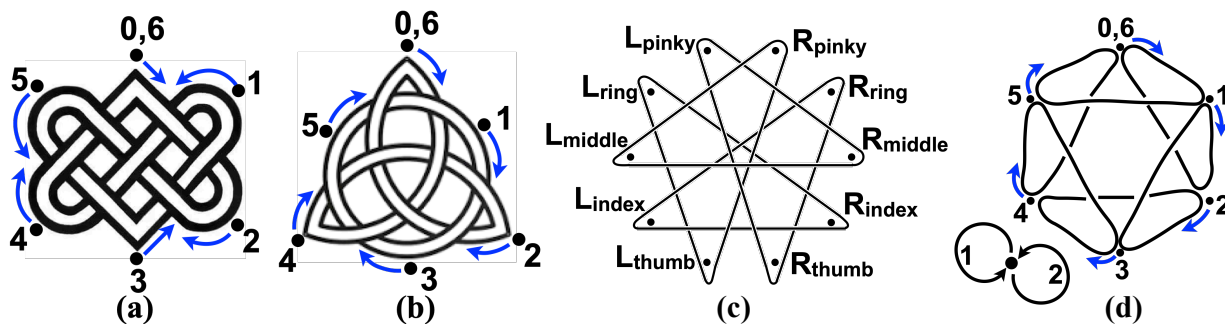


Figure 3: (a) Ball passing pattern Celtic knot $\{6/(2,3,-2,-2, 3,2)\}$; or six dancers following arrows. (b) Six dancers following $\{6/2\}$ and $\{6/-2\}$ patterns, or 2-ball passing. (c) $\{10/(3,5)\}$ string figure. (d) String loop octahedron $2\{6/(1,-2)\}$ and “bouquet” from which it is derived.

Connections can also be made between Celtic and other knots, ball passing (or juggling) patterns, and string figures. Figure 3(a) is a popular Celtic knot sometimes known as a Celtic Love Knot due to the interlocking heart shapes. It could also represent a ball passing pattern for six participants, following the

arrows, of $\{6/(2,3,-2,-2,3,2,)\}$, with the ball starting at position 0. Figure 3(b) is a well-known Celtic knot which is a variation on the Triquetra or Trinity knot. It could represent a ball passing pattern for six participants and two balls, one ball starting at position 0 and following a $\{6/-2\}$ pattern, the other starting at position 1 and following a $\{6/2\}$ pattern. Or it might represent the movements of six pairs of dance partners in which six of the partners remain static in positions 0 through 5, while three of their partners move 2 positions counterclockwise to new partners and three other partners move 2 positions clockwise to new partners. Figure 3(a) could also represent a circular partner dance for six pairs of dancers, with dancers following the arrows to the next partner. In all these designs of dance paths, the under curves precede over curves in time. See [12] for a description of how the undercurve/overcurve features determine how they may be performed, and for the connections to knot theory and other topological concepts.

String figures, designs usually created on the fingers and hands with a small loop of string, have extremely ancient and surprisingly worldwide origins. Figure 3(c) shows a $\{10/(3,5)\}$ design for a string figure, made up of a single unknotted loop, in which the ten fingers of two hands are the vertices. It is meant to duplicate as closely as possible a design by Diana Davis titled "A Celtic-knotted periodic path on a pentagon billiard table" exhibited at the Bridges Math and Art conference 2023, see [5]. This string figure is the unknot rather than Davis's alternating Celtic knot. A video showing its construction is at [14]. Instructions for $\{9/4\}$, $\{7/3\}$, $\{5/2\}$ and $\{11/5\}$ string figures are at [11] along with a video at [4, "Making Stars"]. Figure 3(d) shows a $\{6/(1,-2)\}$ design. It duplicates the design of a three-person string loop octahedron. Each vertex is held by one hand; see a video at [4, "String Quartet"] showing how it is created. This can also be seen as the "derived graph" of the two-petal "bouquet" in the lower left of Figure 3(d) using methods of topological graph theory [6, pg. 60]. In fact the $\{n/A_m\}$ designs can be understood as the application of bouquets with weights assigned in Z/nZ used to derive these designs. Figure 3(d) might also be a design for six dancers who each move through each of the six positions, again in which under curves precede over curves, or for ball passing or juggling patterns. Here are instructions for making the $\{10/(3,5)\}$ string figure:

1. Use a loop of length about 30 inches.
 2. Start with the loop on both palms held by thumb and ring fingers of each hand.
 3. Lt index finger hooks rear ring string from above and shares loop with Rt index.
 4. Lt pinky hooks ear thumb string from above and shares loop with Rt pinky.
 5. Lt middle hooks rear pinky string from above and shares loop with Rt middle.
 6. Adjust figure on hands so loops on fingers are as equidistant as possible.
- (For string figure help consult string figure vocabulary lessons at isfa.org/!)

One further entertaining application of these ideas involves the long history of puzzles centered on finding Hamiltonian tours of rectangular boards by chess pieces that alternate between moves of two pieces, for example between knight and bishop moves [7, 15].

Shirt Lace Buttons



Figure 4: *Shirt lace buttons by Laurel Shastri exhibiting several of the $\{n/(a,b)\}$ patterns.*

The shirt lace buttons created by Laurel Shastri and shown in Figure 4 exhibit some of the $\{n/k\}$ and $\{n/(a,b)\}$ patterns we have been describing above [16]. For example, the third button from the left shows a regular 7-gon together with a $\{7/3\}$ star polygon. But it could also be considered (and constructed) as a $\{7/(1,3)\}$ design since both polygons are built with yellow thread.

Summary and Conclusions

Many art forms utilize designs described here as pseudo-Hamiltonian graph tours, and tabulating more examples and explanations of their connections could make a useful project. Further mathematical explorations of $(a_1, a_2, a_3, \dots, a_m)$ -step k -Hamiltonian tours of the cycle C_n and their applications in various art and movement forms would be a useful and engaging investigation. Knowledge about them could also inform aesthetic decisions in a variety of art forms.

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