Heart of Domain Coloring

Ulrich Reitebuch[1], Henriette Lipschütz[2], Konrad Polthier[3], and Martin Skrodzki[4]

[1]Institut für Mathematik, FU Berlin, Germany; Ulrich.Reitebuch@fu-berlin.de
[2]Institut für Mathematik, FU Berlin, Germany; Henriette.Lipschuetz@fu-berlin.de
[3]Institut für Mathematik, FU Berlin, Germany; Konrad.Polthier@fu-berlin.de
[4]Computer Graphics and Visualization, TU Delft, Netherlands; mail@ms-math-computer.science

Abstract

Domain coloring allows for visualization of complex functions. In this paper, we give a brief introduction to domain coloring in general. Then, we investigate its capabilities to create specific shapes, namely hearts. Finally, we present a gallery of our findings.

Visualizing Functions

The graph of a function connects its domain to its image in a single representation. The situation we are most familiar with illustrates a piece of a continuous function $f : I \subset \mathbb{R} \to \mathbb{R}$ over some interval I of the real numbers in the plane. The visualization of two one-dimensional objects like the interval and the values of the function over the interval is possible in a two-dimensional space. Increasing the number of parameters in the domain to two gives a function $g : \mathbb{R}^2 \to \mathbb{R}$ which can be drawn in various ways. The most common ones are to go to three-dimensional space and consider all those points in the x-y-plane belonging to the domain. These are then equipped with a height value according to the function value. This leads to a segment of a curved surface as depicted in Figure 1. An alternative to this approach is to stick with a two-dimensional representation for the domain and to show the function values via a color map as done in weather forecasts or contour lines like those shown on a hiking map. In the latter, the density of the contour lines intuitively illustrates the extent of the slope in a small neighborhood around a point, as displayed in Figure 1.

Now, turning to complex functions $f : \mathbb{C} \to \mathbb{C}$, we increase the number of components in the image space by one, as complex numbers can be represented in the Gauss plane, where each point (x, y) represents the real and imaginary part of a complex number $z = x + iy$. As both the domain and the image space of f are thus two-dimensional, visualizing f poses the challenge of visualizing a four-dimensional object.

Several approaches to visualize complex functions were already introduced to the Bridges community. For instance, Anne Burns interprets complex functions as vector fields over their domain [3]. Similar to the approach discussed here, this allows an easy recognition of singularities and their multiplicities. Konstantin

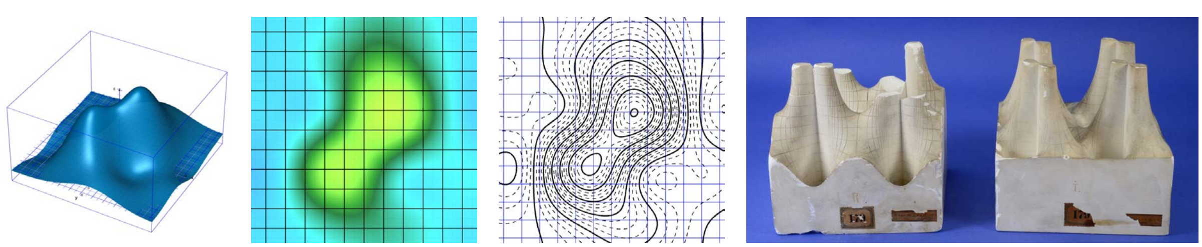


Figure 1: *From left to right: Plots of a function $f : \mathbb{R}^2 \to \mathbb{R}$ as a curved surface in $\mathbb{R}^3$, as a color map, and as level sets; plaster model of real and imaginary part of Weierstrass elliptic function [15].*

Poelke and his co-authors utilize a specific property of complex analysis, the Schwarz reflection principle, to create *complex mandalas* [8]. Their visualizations of complex polynomials connect to the set of zeros of these polynomials. Frank A. Farris employs complex Fourier series on top of a domain coloring approach [5]. Applying this to photos with varaying degrees of color symmetry illustrates various conjugate symmetry types. Finally, Caroline Bowen gives complex functions a physical reality by creating 3D contour plots from paper cut-outs [2]. The height structure of the sculptures reveals a certain evolution of the plots. In the spirit of these works, we will investigate the relationship between a complex function and its visualization.

In the next section, we explain one way to capture all the information needed to tackle the challenge of representing a complex function f via a suitable coloring scheme of the complex plane. While alternative visualizations exist, for instance via Riemann surfaces, we focus on this visualization that is representable on two-dimensional paper. Therefore, we proceed to consider three specific types of complex functions in detail in order to illustrate the concept of domain coloring. Finally, we present our investigations of how to create visualizations of heart-shaped regions via domain coloring and show a collection of the results in a gallery.

From Complex Functions to Domain Coloring and Back

There are various ways to visualize complex functions. While in the end of the 19th century representations were three-dimensional objects made from plaster, see Figure 1, the colorful images present nowadays have roots in the 1990s [6]. The term *domain coloring* was coined by Farris in 1998 [4]. We will follow an implementation-oriented approach [7]. Let $f : \Omega \subset \mathbb{C} \to \Xi \subset \mathbb{C}$ be a complex function. We assume the domain Ω and the image $f(\Omega) = \Xi$ to be topological disks, presented throughout the paper as rectangles. Furthermore, we consider a color map $c : \Xi \to \text{RGB}$, assigning each value in the image of f a color in the *RGB* color space, where every element is represented as a weighted combination of *R*ed, *G*reen, and *B*lue.

To color the domain based on f, we perform the following steps. First, the domain Ω is discretized into small squares, called *pixels*. Every pixel i is then identified with its center point z_i in $\mathbb{C}$. Second, for each pixel i, we obtain a color value by evaluating $c \circ f$ on z_i, hence, its color will be given by $c\,(f\,(z_i))$. The steps are illustrated in Figure 2. There are similar approaches known like the *phase portraits* by Elias Wegert [9] which display $f(z)/|f(z)|$ instead of $f(z)$ and solely require a one-dimensional color space in shape of a circle.

In this paper, we focus on specific complex functions, namely polynomials, rational functions, and trigonometric functions. The total quantity of these functions will be denoted by $\mathcal{F}$. A *polynomial* is a function $p : \mathbb{C} \to \mathbb{C},\ z \mapsto a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, where $n \in \mathbb{N}_0$ and $a_j \in \mathbb{C}$ for all $j \in \{0, \ldots, n\}$. A *rational function* is the quotient of two polynomials p and $\tilde{p}$, that is $r(z) = p(z)/\tilde{p}(z)$, where $\tilde{p} \not\equiv 0$. We call $z_0 \in \mathbb{C}$ a *zero* of $f \in \mathcal{F}$, if $f\,(z_0) = 0$. A *pole* of a function $f \in \mathcal{F}$ is a zero of $1/f$. If $f \in \mathcal{F}$ is a polynomial or a rational function, its zeros and poles can be characterized further by their *order*. The order is defined as the multiplicity of the zero or pole, respectively, and can be read off from a factorization of f,

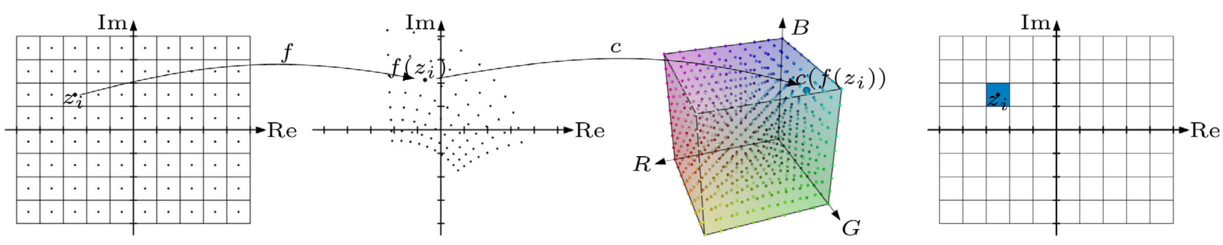


Figure 2: *From left to right: A pixel-center z_i is mapped to its image $f(z_i)$. Subsequently, $f(z_i)$ is mapped to the RGB color space. The last image shows the pixel i in the domain correspondingly colored.*

which is given by $f(z) = c \cdot (z - c_1)^{m_1} \cdot \ldots \cdot (z - c_k)^{m_k}$, where $m_1, \ldots, m_k \in \mathbb{Z} \setminus \{0\}$ and $c \in \mathbb{C}$. Here, the complex numbers $c_1, \ldots, c_k$ are the zeros or poles of f and $|m_1|, \ldots, |m_k| \in \mathbb{N}$ are their respective orders. This is completely analogous to the factorization of a polynomial in the reals. The *trigonometric functions* can locally be approximated sufficiently well by polynomials such that the definition of a pole can be transferred to functions like $1/\sin(z)$. According to the polynomial approximation, the zeros of sine and cosine are of order 1. We restrict our considerations to these three types of functions because representatives of said classes will later be used to create the heart shapes we aim at. As the zeros and poles of a complex function carry such important information, we would like the chosen color scheme to emphasize these characteristics.

Having introduced how a single pixel is colored, we consider a small collection of functions from $\mathcal{F}$ to understand how they are visualized via this coloring. A property of the color function c we would like to have is injectivity such that the value of $f(z)$ can be uniquely read off from the color. One possible color scheme uses a polar coordinates representation [7]. It maps the angle to a color wheel, where the radius encodes the saturation of the color. Displaying the saturation comes with the drawback that black, representing 0, and white, representing values approaching (or attaining, see example $1/z$ below) ∞, dominate such that the angle value is hard to read. Instead of using saturation for the radius, we draw integer grid lines to illustrate the distance to the origin, see the leftmost image in Figure 3.

For better readability, we color the real and imaginary axes in black and all integer grid lines in white. All other complex numbers are mapped to different colors by their angle to the positive real axis, using four color spectra, one for each quadrant. In order to avoid the combination of red and green within the same image to make the graphics enjoyable for color-blind people, we included solely the colors yellow, green, and blue into the color wheel. In the upcoming images, zeros are indicated by green dots while the poles are shown in blue. The domain coloring of the identity function $z \mapsto z$ shows the chosen color map c, which is illustrated in the leftmost image in Figure 3.

After having investigated the identity function, we increase the power of z to 2 and consider the polynomial $z \mapsto z^2$. This polynomial has a single zero $c_0 = 0$ of order 2, which is indicated by the green dot in the center of the second image from the left in Figure 3. Since squaring a complex number can be done by squaring the radius and multiplying the polar angle by 2, the complete color wheel is traversed around 0 twice, in the same order as shown in the leftmost image in Figure 3. This causes the white parameter lines to not appear straight anymore. In general, every monomial z^n, $n \in \mathbb{N}$, shows a similar behavior, traversing the complete color wheel n-times around its zero of order n.

As a next example, we now turn to $f(z) = 1/z$. Earlier, we defined a pole as a zero of $1/f$ which is not a helpful definition when it comes to visualizing the situation. For the functions in $\mathcal{F}$, it is reasonable to allow ∞ as its evaluation and to show it in black. This can be made mathematically rigorous by a compactification of the complex plane, leading to the Riemann sphere, see [1, Sec. 1.5] for details. Therefore, we can do the standard calculations with ∞ in a natural way. Hence, considerations similar to those made in the real case allow us to detect ∞ as a zero of order 1 of $1/z$. For any complex number $z \in \mathbb{C}$, the expression $1/z$ can be rewritten as $\bar{z}/|z|^2$, with $\bar{z}$ the *complex conjugate*, which allows for the description of mapping z to $1/z$

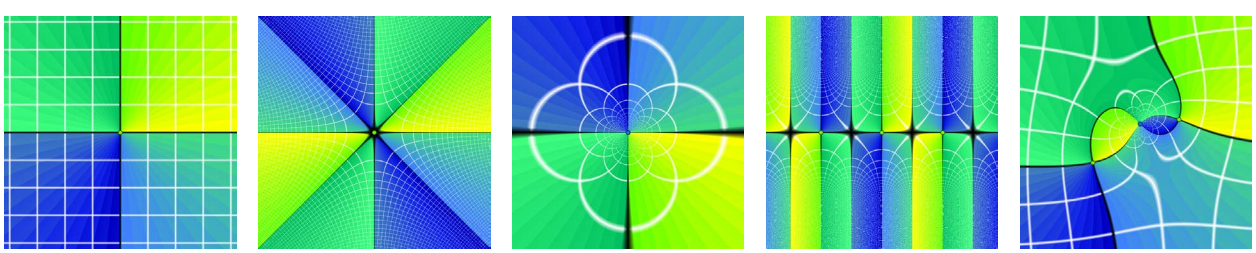


Figure 3: *From left to right:* $z \mapsto z$, $z \mapsto z^2$, $z \mapsto 1/z$, $z \mapsto \sin(z)$, *and* $z \mapsto z \cdot (z - 2 - i)/(z - 1.1 - 0.9i)$.

as the inversion on the unit circle combined with the reflection on the real axis. Therefore, we expect the corresponding domain coloring to show the outer part of the color wheel around the origin in reversed order. This is exactly what we observe in the middle image of Figure 3. Further, the white grid lines that were equally distributed before are now mapped to circles by the inversion on the unit circle. They also appear denser towards the origin since the inversion on the circle interchanges the locations of 0 and ∞. Similar to the monomial discussed above, the rational function $1/z^n$ also traverses the complete color wheel n-times in reversed order.

Next, we will consider the sine function. This function has infinitely many real zeros $c_n = n \cdot \pi$ for $n \in \mathbb{Z}$, each of order 1. By the considerations above, there are infinitely many copies of the approximated color wheel placed along the real axis, as can be observed in the second image from the right in Figure 3. At those points where the real sine function has a minimum or a maximum, the black grid lines branch. Hence, there are regions arising where two color spectra lie next to each other but do not do so in the original color wheel.

Next, we consider the function $z \mapsto z \cdot (z - 2 - i)/(z - 1.1 - 0.9i)$, after having considered comparably simple functions before. Based on the definitions given above, the function has two zeros of order 1—$c_1 = 0$ and $c_2 = 2 + i$—and a single pole $c_3 = 1.1 + 0.9i$ of order 1. These three are not placed as symmetrically as those of the examples considered so far. Hence, the grid lines represented in black do not appear as straight lines anymore, see the rightmost image in Figure 3.

After having introduced how a given function is visualized, we might ask for the other way around. Which information can we read off from a given domain coloring without knowing the visualized function explicitly? In fact, if we focus on a point $z \in \mathbb{C}$ and if the complex function f is *analytic* at z—differentiable in a neighborhood of z—, then it is uniquely determined by its values in an ε-neighborhood around z, [1, Corollary 6.10]. Thus, in an idealized version of domain coloring with an infinite amount of pixels, an infinitely small open neighborhood suffices to reconstruct f. Yet, for the classes of polynomials and rational functions, even the discretized version of domain coloring provides ample information on the functions. In the previous section, we encountered the duplication of the color wheel's colors while increasing the order of the zeros of the polynomial considered. Hence, counting the number of times the colors from the color wheel are placed around a single point gives us some information on the orders of zeros of the considered polynomial. Analogously, we can find the order of a function's poles.

The Heart of Domain Coloring

Nowadays, the shape of a heart represents the organ itself and metonymically kindness and (romantic) love. In the medieval ages, the human heart was seen as the source of emotions and being in opposition to cognition and reason. This is still present in the Spanish words *razón* (mind, reason) and *corazón* (heart or "which is existing next to the mind"). The usage of this form goes back to approximately 3,000 BC, where it originated

Figure 4: *Collection of hearts from different contexts. From left to right:* The Offering of the Heart*, flemish tapestry, early 15th century [11]; hand guard (*tsuba*) of a Japanese sword, 16th century [10]; bleeding heart,* Lamprocapnos spectabilis*, image by A. J. Wendel, 1868 [12]; coat of arms of Weimar, Germany [13]; Luther rose [14].*

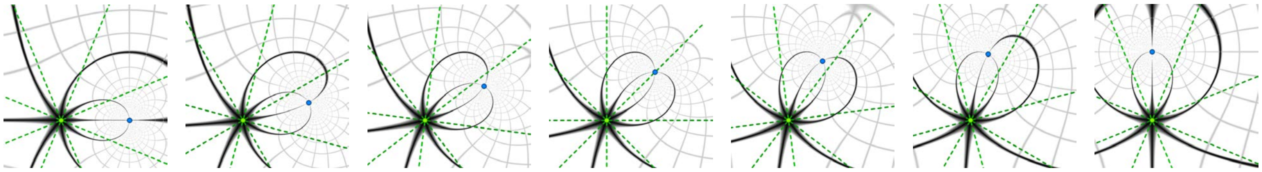


Figure 5: *Parameter lines for zero of order* 2 *and pole of order* 1.

as an abstraction of fig- or ivy leaves—with further examples from flora known like the bleeding hearts, shown in the center of Figure 4. In Europe, the heart became popular in the medieval ages, where it appeared first as a fig leaf colored in red as a symbol of love, as illustrated in the leftmost image in Figure 4. This shape can be found in other ages and cultural settings as well. Since ancient times in Japan, the heart symbol has been called *Inome*, meaning "the eye of a wild boar", and it was displayed on weapons like on the hand guard of a sword as shown in the second image from the left in Figure 4. Here, this shape refers to the bravery and the determination with which an attacked boar would attack its enemy. Hearts are also found in heraldry, see second image from the right in Figure 4, where they originally referred to the leaves of water-lilies. The symbolism changed over time from referring to nature to the usage of the heart as a symbol of love. Further, hearts are depicted often in ecclesiastical surroundings, like in the protestant church where it is incorporated into the Luther rose, shown in the rightmost image in Figure 4.

Since a heart is made from two curved arcs that are reflection symmetric, this omnipresent symbol compels because of its simplicity and symmetry from a mathematical point of view. After having introduced the basic properties of domain coloring, we will use our gained knowledge to create specific shapes—heart shapes. We collect functions with a distribution of poles and zeros such that the contour of the heart is solely made from points with either imaginary or real function values. The contour of a heart as shown in Figure 4 has an additional engaging property as it has two sharp corners both lying on the symmetry axis. As visible in the three middle images of Figure 3, zeros and poles create sharp corners with those lines containing pure real and imaginary values. Therefore, we use zeros or poles to create the sharp corners of the heart. In order to guarantee the correct grid lines to run through the zeros or poles, they have to be carefully adjusted. Here, we assume that each of the four quadrants is equipped with its own color spectrum, which will be enclosed by the black grid lines after mapping the color wheel under some suitable function to the plane. Further, we saw that the placement of poles and zeros causes the black grid lines to curl and cross. In the following, we make use of these degrees of freedom and create several classes of hearts from them.

We will emphasize zeros with green dots and poles with blue ones and use dashed green lines to show the angle bisector directions between parameter lines in the zero. In the first example, shown in Figure 5, to create the convex vertex or tip of the heart, we place a zero of order 2 there. To create the convex corner we place a pole of order 1 with distance 1 to the zero. If the pole is moved on a circle around the zero, at some point the pole will be exactly on one of the dashed bisector lines and we get a symmetric heart shape, using three areas enclosed by black parameter lines connecting the zero to the pole, shown in the center of Figure 5.

In the second example, shown in Figure 6, we create two linked hearts, sharing a concave corner. At this common concave corner and at the two tips of the hearts we place zeros of order 1. At the two crossing

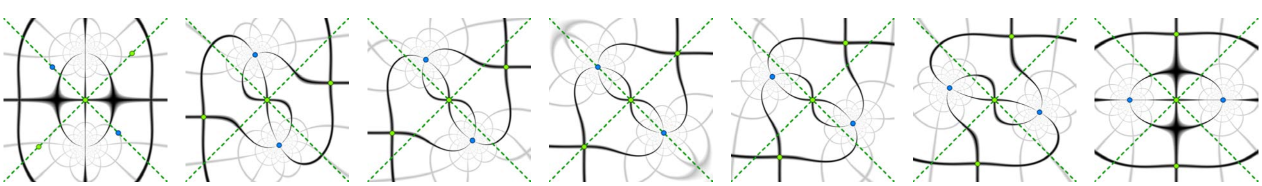


Figure 6: *Parameter lines for three zeros of order* 1 *and two poles of order* 1.

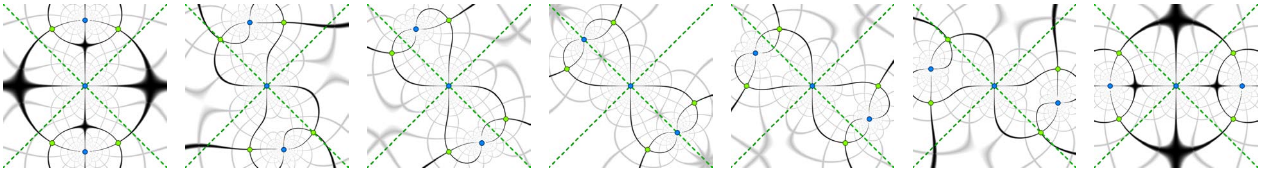


Figure 7: *Parameter lines for four zeros of order* 1 *and three poles of order* 1.

points of the two heart shapes we place two poles of order 1. Angle bisector lines between the parameter line directions in the common concave corner of the hearts are again shown by green dashed lines. The configuration is created symmetrically to the common concave corner of both hearts. When the points are rotated around the center point, in the configuration shown in the middle image of Figure 6, the two green tip points and the two blue points are positioned on the angle bisector lines between the parameter lines in the center point. For this configuration, we get two linked heart shapes.

The third configuration, shown in Figure 7, creates two hearts touching at the tip. Here, we place poles of order 1 at all corners of the two hearts and add to each of the heart shapes two zeros of order 1. These are used to let the inner parameter lines at the concave corners cross through the contours of the hearts. For the configuration in which the poles for the concave corners are placed on the dashed angle bisector lines, the image becomes symmetric and creates two symmetric hearts touching at the tip, as shown in the middle image of Figure 7.

The fourth and final configuration, shown in Figure 8, consists of a zero of order 2 for the tip of the heart, a zero of order 1 inside the heart shape, and a pole of order 2 for the concave corner. Here, we get two types of heart shapes, one for the symmetric configuration shown in the middle image of Figure 8, and a different one for the symmetric configurations in the leftmost and rightmost image. For the configuration in the middle, the concave corner lies on a green dashed line. For the configuration in the leftmost and rightmost image, the concave corner lies on a yellow dashed line, showing the parameter line directions at the heart. This is also the angle bisector direction between the two neighboring parameter lines. In both cases, we get a symmetric heart shape, but the angles at the corners are different for the two types of configurations.

These four configurations are creating heart shapes based on the domain coloring principles, yet solely focusing on the axes. This is because the black axes make for the most striking visual impressions. However, the full potential of domain coloring is only unleashed when choosing a suitable color map c. Hence, in Figure 9, we combine the configurations described above with the color scheme from Figure 3. The four used configurations have in common that the orders of zeros in sum exceed the orders of poles by one. Therefore, for each of these configrations, the grid lines approximate a regular square grid with growing distance from the heart region. A good balance between orders of zeros and poles makes it easier to have several heart shapes in one image. At this point, we do not claim completeness, but aim with this paper at serving as a showcase of the potential of domain coloring in creating specific—in this case heart—shapes. We hope that this inspires other mathematical artists to come up with their own shape representations via domain coloring. We encourage the reader to create individual pieces, e.g. via [16].

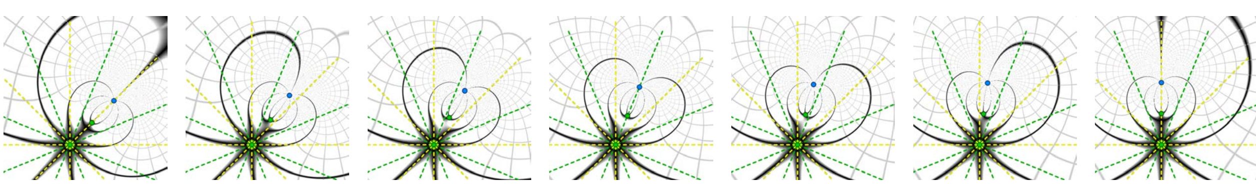
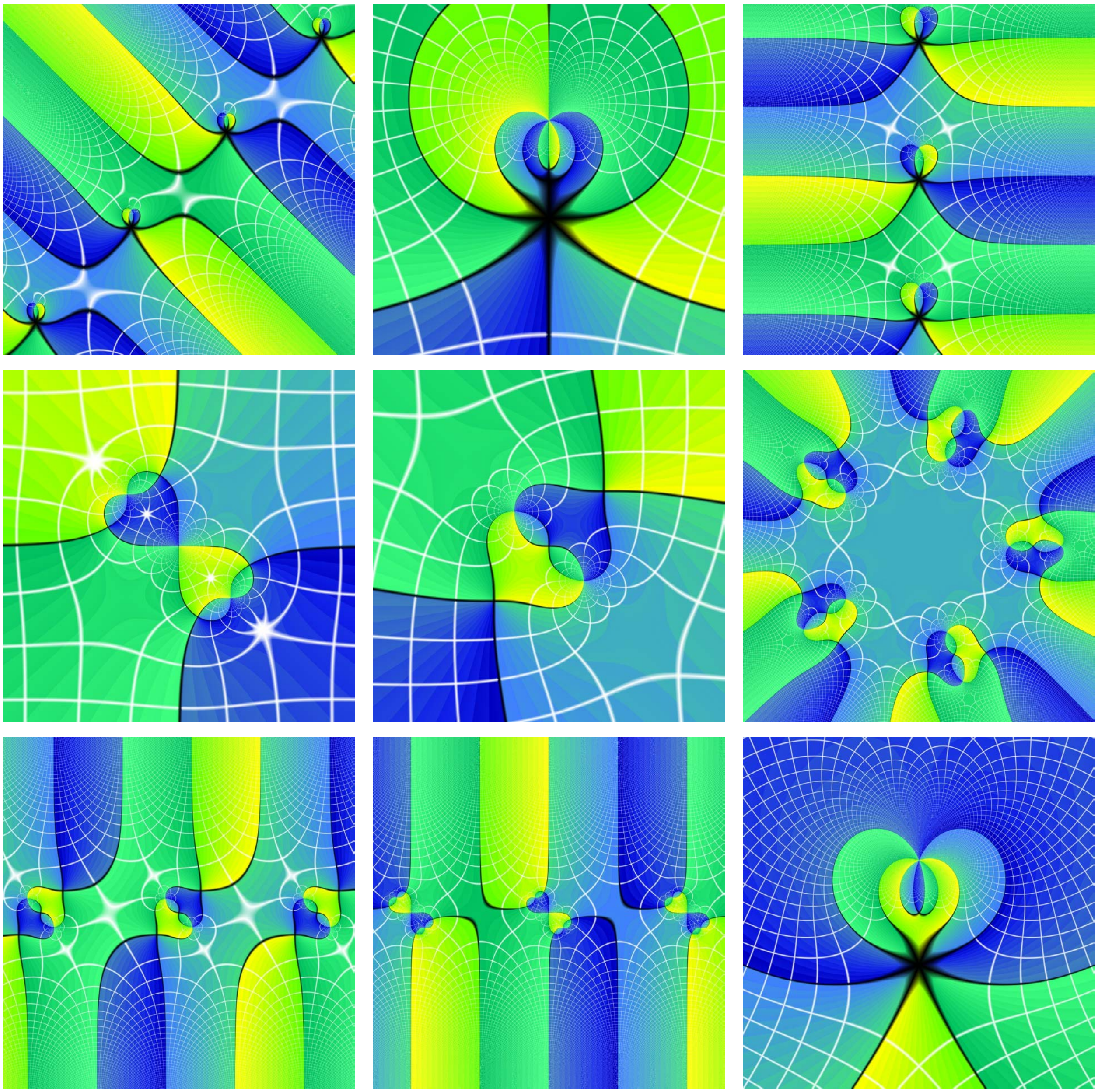


Figure 8: *Parameter lines for zeros of order* 1 *and* 2 *and one pole of order* 2.



$$f_1(z) = \frac{i\cdot\cos\big((1-i)^2\cdot z^2\big)}{\cos((1-i)\cdot(z-0.28i))}, \qquad f_2(z) = \frac{z\cdot(z+i)^2}{(z-i)^2}, \qquad f_3(z) = \frac{(-1.1+1.1i)\cdot\sin^2(iz)}{\sin\Big(iz+\frac{\sqrt{2}}{2}\Big)},$$

$$f_4(z) = \frac{\sqrt{2}\cdot\big(1-iz^2-z^4\big)}{z\cdot\big(1-i\cdot z^2\big)}, \qquad f_5(z) = \frac{7\cdot(z+1+i)\cdot(z-1-i)}{z^2+0.8i}, \qquad f_6(z) = \frac{(-1+i)\cdot\big(\frac{3}{2}z^{10}-2z^5+6\big)\cdot\big(z^5-1.2\big)}{4\cdot\big(\frac{1}{2}z^{10}-2z^5+0.7\big)},$$

$$f_7(z) = \frac{(1+0.3i)\cdot\sin(z)\cdot\sin\big(z+\frac{1}{2}+\frac{1}{2}i\big)}{\sin^2(z)+0.2i}, \qquad f_8(z) = \frac{\sqrt{2}\cdot\big(1-4i\cdot\sin^2(z)-16\cdot\sin^4(z)\big)}{2\cdot\sin(z)-9i\cdot\sin^3(z)}, \qquad f_9(z) = -\frac{z^2\cdot(1-i)^2\cdot(z\cdot(1-i)-1-i)}{(z\cdot(1-i)-2\cdot(1+i))^2}.$$

Figure 9: *Gallery of heart functions. Below, the visualized functions are displayed from top left to bottom right analogously to the images.*

Summary and Conclusion

In this paper, we gave a brief introduction to domain coloring to visualize complex functions. We based our investigations on polynomials, rational functions, and trigonometric functions and used their characteristics to illustrate the process of domain coloring. After motivating the heart shape to be of cultural relevance, we used said set of complex functions to create heart-shaped regions within their domain coloring.

Acknowledgments

We thank K. Poelke for the effort he spent on making domain coloring accessible.

References

[1] J. Bak and D. J. Newman. *Complex Analysis*, 3rd ed., Springer, 2010.

[2] C. Bowen. "Cut Colored Paper Sculptures of 3D Contour Plots of the Real and Imaginary Parts of Complex Functions." *Bridges Conference Proceedings*, Waterloo, Canada, Jul. 27–31, 2017, pp. 375–378. https://archive.bridgesmathart.org/2017/bridges2017-375.html.

[3] A. Burns. "The Art of Complex Flow Diagrams." *Bridges Conference Proceedings*, Coimbra, Portugal, Jul. 27–31, 2011, pp. 51–58. https://archive.bridgesmathart.org/2011/bridges2011-51.html.

[4] F. A. Farris. "Review of Visual Complex Analysis, by Tristan Needham." *American Mathematical Monthly* 105 (1998), No. 6: 570–576.

[5] F. A. Farris. "Natural Color Symmetry." *Bridges Conference Proceedings*, Waterloo, Canada, Jul. 27–31, 2017, pp. 131–138. https://archive.bridgesmathart.org/2017/bridges2017-131.html.

[6] D. Kucerovsky. "An algorithm for the visual representation of a two-dimensional vector field." *Conference Record of the 1990 IEEE Industry Applications Society Annual Meeting*, 1990, pp. 903–909.

[7] K. Poelke and K. Polthier. "Domain Coloring of Complex Functions: An Implementation-Oriented Introduction." *IEEE Computer Graphics and Applications*, vol. 32, no. 5, 2012, pp. 90–97.

[8] K. Poelke, Z. Tokoutsi, and K. Polthier. "Complex Polynomial Mandalas and their Symmetries." *Bridges Conference Proceedings*, Seoul, Korea, Aug. 14–19, 2014, pp. 433–436. https://archive.bridgesmathart.org/2014/bridges2014-433.html.

[9] E. Wegert. "Visual Complex Functions.", 1st ed., Springer, 2012.

[10] Image taken from https://commons.wikimedia.org/wiki/File:Tsuba-Ethno_BHM_1922.266.1836-P8260052-white.jpg.

[11] Image taken from https://commons.wikimedia.org/wiki/File:Tapestry_by_unknown_weaver_-_The_Offering_of_the_Heart_-_WGA24173.jpg.

[12] Image taken from https://commons.wikimedia.org/wiki/File:WitteHeinrichFlora1868-016-Lamprocapnos_spectabilis.png.

[13] Image taken from https://de.wikipedia.org/wiki/Datei:Wappen_Weimar.png.

[14] Image taken from https://commons.wikimedia.org/wiki/File:Luther_Rose.png.

[15] Image taken from https://americanhistory.si.edu/collections/nmah_694015.

[16] Interface available at https://www.desmos.com/calculator/1jminz56uo.