Using Origami to Make Abstract Elevations of Polyhedra

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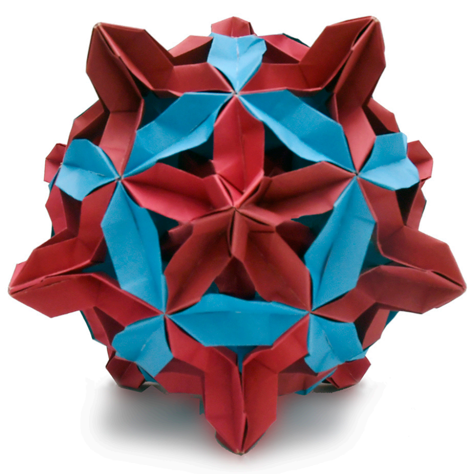
Abstract

This workshop introduces two new modular origami units that I designed. We will learn how to fold and join these modules. Then we will use these units to make origami sculptures inspired by elevations of polyhedra. We will also discuss other geometric sculptures that can be made with these units.

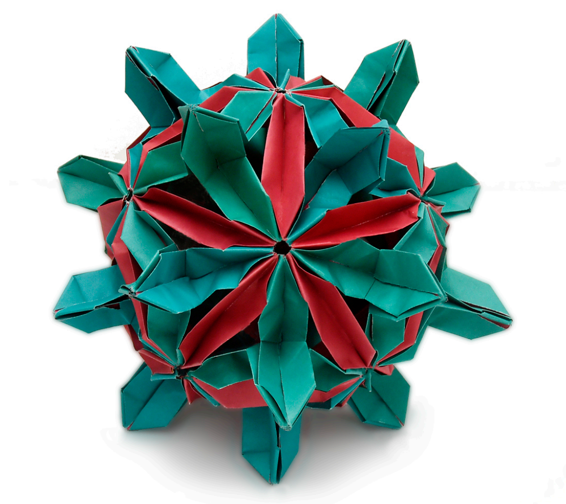
Introduction

Figure 1 shows two geometric sculptures I designed that were inspired by elevations of polyhedra. I named these pieces "Elevated Dodecahedron" and "Elevated Icosahedron." I used quotes in these names, as the geometric sculptures are not exact elevations but are abstractions of this concept. An elevation of a polyhedron is formed by adding an n-sided pyramid on top of every face of a polyhedron, where n is the number of edges of the corresponding face [5]. The faces of the pyramids are all equilateral triangles.

These sculptures are constructed from origami modules which I call A-units and B-units. Consider Figure 1(a). The blue A-units form a dodecahedron, while the points at the ends of each group of five red B-units correspond to the vertices of a pyramid with a pentagonal base. Similarly, Figure 1(b) corresponds to an elevation of an icosahedron. The red A-units form an icosahedron, and the turquoise B-units correspond to the triangular pyramids on the 20 faces.



(a)



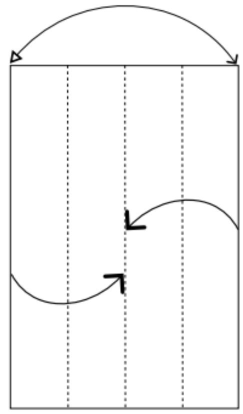
(b)

Figure 1: *(a) "Elevated Dodecahedron" and (b) "Elevated Icosahedron."*

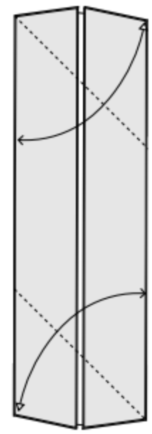
In this workshop, we will first learn how to fold the units for these sculptures. We will then learn how the modules have been carefully designed so that the distance between the endpoints of both types of units are the same and thus correspond to the vertices of an elevation. Next, we will work together to construct the models shown above. Finally, we will discuss how these units can be used to construct other geometric sculptures.

A-Unit Instructions

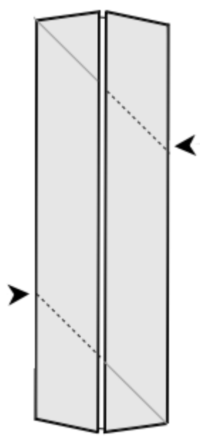
For these units, we will use paper that is 3 inches wide and 5.2 inches high. The mathematics for calculating the height can be found in the next section. Any size paper with the same proportions can be used.



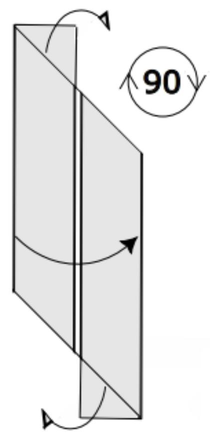
Step 1: Fold in half, then unfold.
Next, fold into 1/4.



Step 2: Fold corners, then unfold.

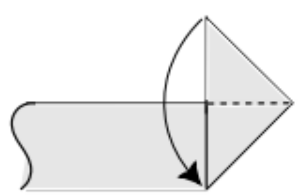


Step 3: Reverse-fold the dotted line.
Repeat on both sides.



Step 4: Turn the outer folds in.
Repeat on both sides, then fold in half vertically. Next, turn the paper 90 degrees.

Step 5: Fold the corner up triangularly.
Repeat on both sides.



Step 6: Fold the corner down.
Repeat on both sides.

Figure 2: *Instructions for A-units.*

B-Unit Instructions

The B-units are made from 3 inches by 6 inches paper. Any size paper with similar proportions will work.

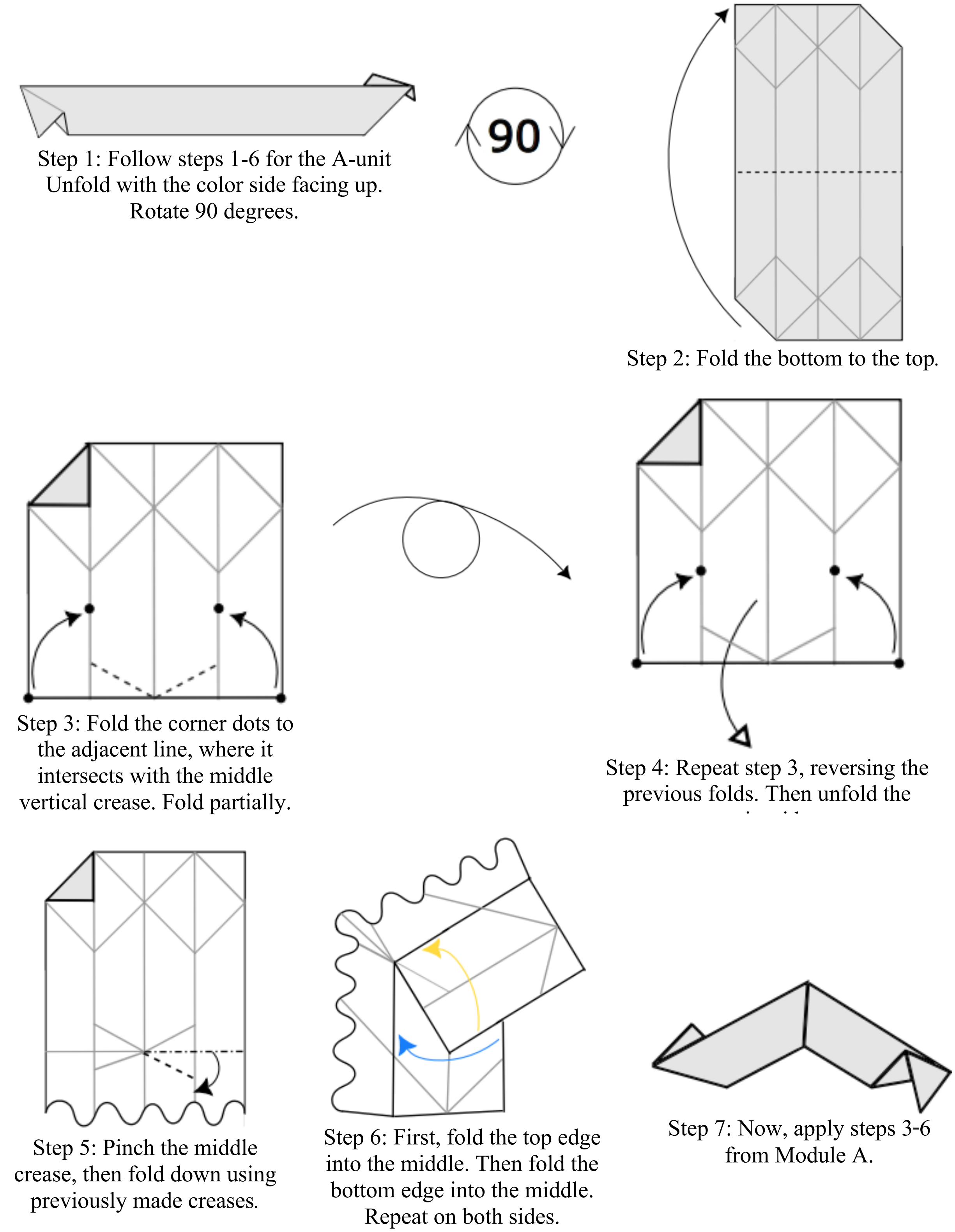


Figure 3: *Instructions for B-units.*

Module Measurements

Since the faces of the pyramids of an elevation are equilateral triangles, the modules for these sculptures have been carefully designed so that the triangles formed by the imaginary lines between the endpoints of the units will be equilateral (see Figure 4(a)).

Consider the image in Step 2 of Figure 3. Since the paper is 6 inches long and 3 inches wide, the length of the green line in Figure 4(b) is $(6 - 2(\frac{3}{4})) = 4.5$ inches. The fold in Step 3 of Figure 3 forms a crease at a 30° angle to the bottom of the diagram since the green triangle in Figure 4(c) is equilateral. The fold in Step 6 forms a 120° angle (Figure 5(a)) since this fold removes 60° from the straight edge of the module.

We will calculate the length of the line between the endpoints of the B-unit when the model is assembled. This line is the yellow side of the equilateral triangle in Figure 4(a). We will then use this length to determine the height, h, of the paper we will use to make the A-units. We want the length of the A-units, shown in pink in Figure 4(a), to be the same length as the yellow sides of the triangle. Thus, the sculptures will be accurate abstractions of the elevations as intended.

When the B-units are closed flat, as in Figure 5(a), the unit has a 120° angle. Each side of the folded unit is ½ (4.5) = 2.25 inches. So, by the law of sines, the distance between the endpoints is $(2.25)\,\frac{\sin(120°)}{\sin(30°)} \approx 3.9$ inches. But when the modules are used in a model, they are not flat, they are opened at angles determined by the geometry of the elevated polyhedron we are modeling.

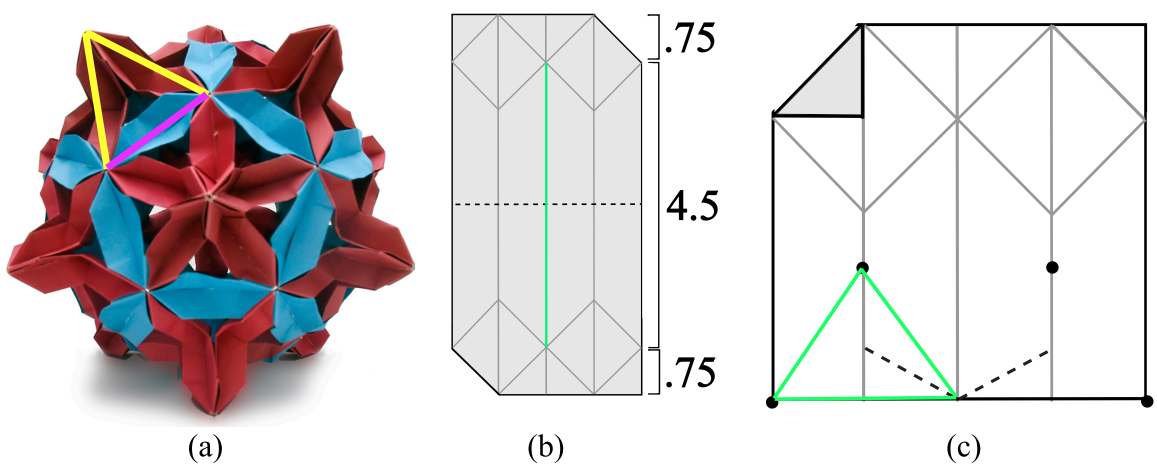
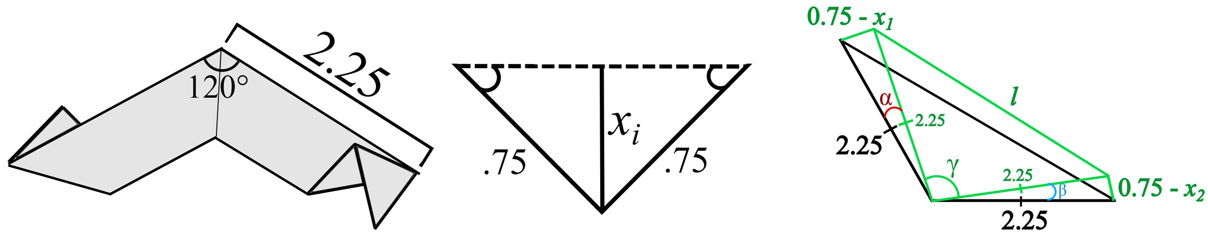


Figure 4: *(a) "Elevated Dodecahedron" with the sides of one elevation triangle outlined in yellow and pink. (b) length of B-unit, (c) equilateral triangle related to the fold in Step 6 of Figure 3.*

Figure 5(b) represents the end view of an open module. This opening angle is dependent on where in the model the end of the unit is placed. Let n be the number of edges for each face of the base polyhedron. If the end of the unit is at the peak of an elevation pyramid, then n units will meet at that point, and the opening angle will be $(360/n)°$. In this case, the height of the triangle in Figure 5(b) will be $x_1 = .75 \cos\left(\frac{180°}{n}\right)$. If the end of the unit is at a vertex where m faces of the base polyhedron meet, then m A-units and m B-units will meet at that point. We will simplify this calculation by assuming the A-units and B-units all open the same amount and this opening angle is $(360/2m)°$. The angled end of a module has length $\sqrt{2(.75^2)}$. At this end, half the dashed line in Figure 5(b) is $\sqrt{2(.75^2)}\ \sin\left(\frac{180°}{2m}\right)$. Hence, the height of the triangle in Figure 5(b) at this end is $x_2 = \sqrt{(.75^2) - 2(.75^2)\sin^2\left(\frac{180°}{2m}\right)}$.



(a) (b) (c)

Figure 5: *(a) Measurements for a closed B-unit. (b) End view of an open B-unit. (c) Measurements for an open B-unit showing angle changes.*

Next, we will calculate how the 120° angle of a closed B-unit will change as the ends of the unit open. As the B-units open, the depth of the side view of the unit will decrease by $0.75 - x_1$ at one end and $0.75 - x_2$ at the other end. Figure 5(c) is a simplified diagram of this side view showing angles α and β, which are the amount by which the 120° angle decreases at each end. By the law of cosines, we get $\alpha = \arccos\left(\frac{(2.25)^2+(2.25)^2-(0.75-x_1)^2}{2\,(2.25)(2.25)}\right)$ and $\beta = \arccos\left(\frac{(2.25)^2+(2.25)^2-(0.75-x_2)^2}{2\,(2.25)(2.25)}\right)$. Thus, the open unit has an angle of $\gamma = 120° - (\alpha + \beta)$. Finally, by the law of sines, the distance between the two endpoints of this unit is $l = 2.25\,\frac{\sin(\gamma)}{\sin(\delta)}$, where the two other angles in this isosceles triangle are $\delta = \frac{180° - \gamma}{2}$. Since we want the A-units to have this same distance *l* between the endpoints, we should use rectangles that are 3 inches by *h* = (*l* + 1.5) inches for the A-units.

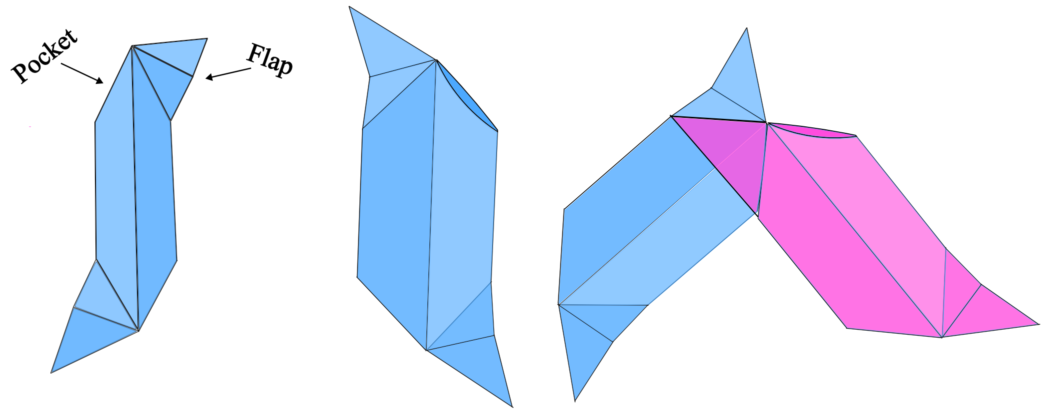
The values for the calculations in this section are given in Table 1. Something interesting and helpful to note for making these two models is that the value of *h* is almost identical for both models. Even though the geometry is different and the ends of the modules open differently in each model, the value of the angle γ is very close in the two structures. Hence, we can use paper that is 3 inches by 5.2 inches to make the A-units for both sculptures.

Table 1: *Calculated values of variables used to determine the paper size for the A-units.*

Elevated Polyhedron	Edges per face (n)	Edges per vertex (m)	x_1	x_2	α	β	γ	δ	l	h
Dodecahedron	5	3	.6068	.5303	3.647°	5.597°	110.8°	34.60°	3.704 in	5.204 in
Icosahedron	3	5	.3750	.6746	9.560°	1.920°	108.5°	35.75°	3.652 in	5.152 in

Connecting Units

Each module has two parts for connecting, a flap and a pocket (Figure 6(a)). To connect two modules, open the pocket of one of the modules as shown in the top right of Figure 6(b). Insert the flap of the second module into the pocket of the open module until the crease of flap aligns with the crease of the pocket, as shown in Figure 6(c). In the actual model, the pink flap will not be visible because it is inside the pocket. Use the flap creases as an indicator to stop inserting the module. Then crease the top of the pocket module to secure the two modules together. Note that A-units and B-units have the same size flaps and pockets, so connections between both types of modules are the same.

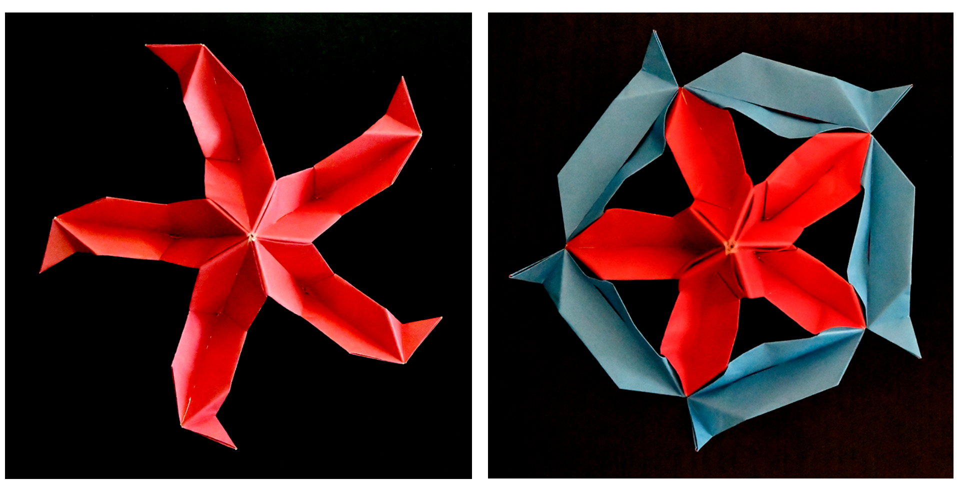


(a) (b) (c)

Figure 6: *(a) B-unit with pocket and flap indicated, (b) backside of a B-unit, (c) connecting another module into the pocket of a module.*

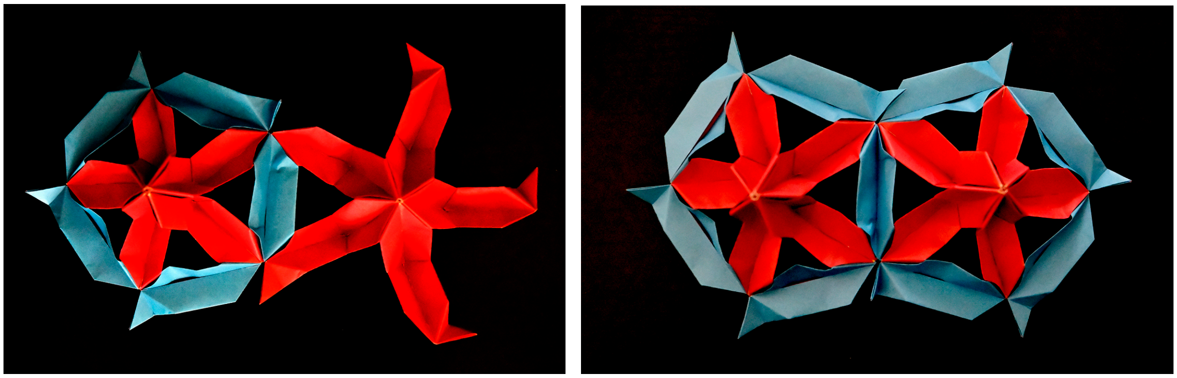
A Sculpture Inspired by the Elevation of the Dodecahedron

The construction of the "Elevated Dodecahedron" requires 30 A-units and 60 B-units. Begin by connecting five B-units in a circular group (Figure 7(a)). Be sure to sharply pinch each crease after connecting units to avoid pieces falling out during construction. We will need 12 of these five B-unit groups. Next, connect one A-unit to two corners between two B-units. Repeat this process four more times around the first group of B-units (Figure 7(b)).



(a) (b)

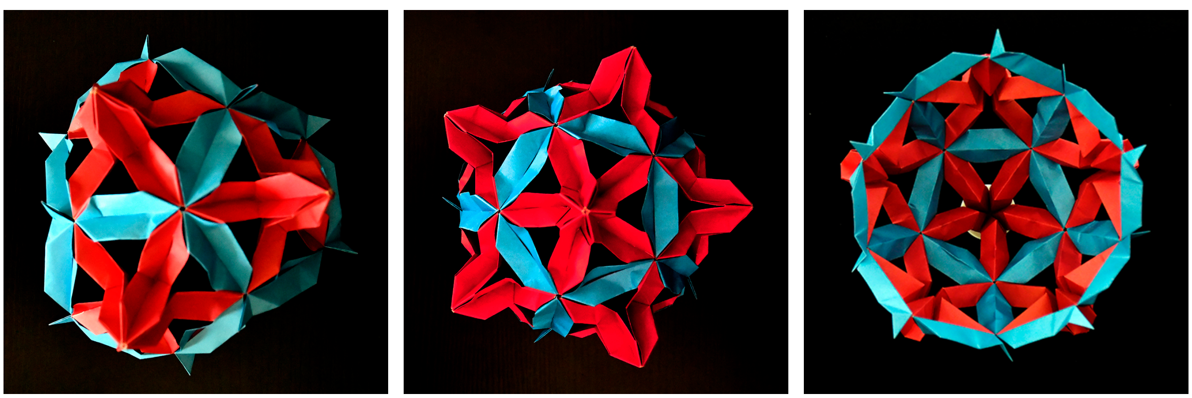
Figure 7: *(a) A group of 5 red B-units, (b) a group of 5 red B-units with blue A-units between B-units.*



(a) (b)

Figure 8: *(a) An additional group of 5 B-units is attached, (b) A-units surrounding both groups of B-units.*

Next, add another group of five B-units to the structure (Figure 8(a)). Then add four more A-units around this group of B-units (Figure 8(b)). Finally, add a third group of B-units to surround the vertex where three A-unit pentagons meet (Figure 9(a)). Continue to add groups of 5 B-units to the existing structure and surround them with A-units to form pentagons. Continue this assembly process until the model resembles Figure 1(a).

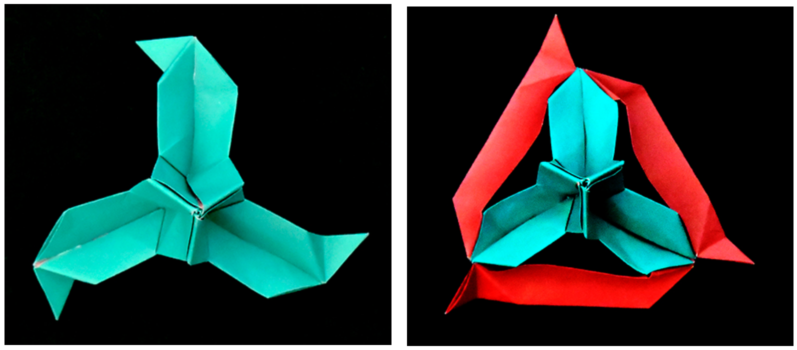


(a) (b) (c)

Figure 9: *(a) 3 groups of 5 B-units, surrounded by A-units, (b) 5 B-units surrounding the central group of B-units, (c) interior view of Figure 9(b).*

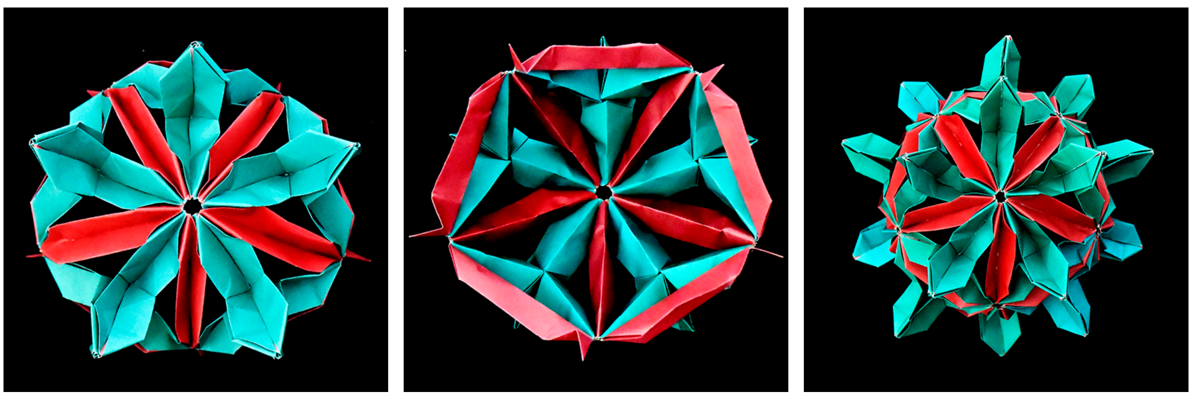
A Sculpture Inspired by the Elevation of the Icosahedron

Figures 10 and 11 show the process of building an "Elevated Icosahedron" (Figure 1(b)). This sculpture requires 30 A-units and 60 B-units. The process is similar to making an "Elevated Dodecahedron," but groups of three B-units will be surrounded by three A-units. Following the geometry of an icosahedron, five A-units and five B-units will meet at each vertex of the A-unit icosahedron. When completing a full circle of modules around a vertex for this model, it may be helpful to partially take out another module to create more room to twist the module in.



(a) (b)

Figure 10: *(a) group of 3 B-units, (b) a group of 3 B-units surrounded by A-units.*



(a) (b) (c)

Figure 11: *(a) 5 groups of three turquoise B-units surrounded by red A-units, (b) flipped view of Figure 11(a), (c) complete "Elevated Icosahedron."*

Further Explorations

I have used these modules to make many sculptures [2][3][4]. The supplement to this paper includes a table that gives the number of units needed to build elevations of various polyhedra.

Acknowledgments

Thank you to Rona Gurkewitz for giving feedback on the instructions and diagrams. Thank you to Eve Torrence for inspiring and mentoring me to complete this paper. The diagrams were created using the Inkscape software application [1].

References

[1] *Inkscape*. https://inkscape.org/

[2] N. Shrestha. "Compound of an Icosahedron and an Icosidodecahedron." *Bridges Aalto 2022 Art Exhibition Catalog*, http://gallery.bridgesmathart.org/exhibitions/2022-bridges-conference/realneel

[3] N. Shrestha. "Compound of a Rhombicosidodecahedron and an Icosidodecahedron." *2023 Joint Mathematics Meetings Art Exhibition Catalog*, http://gallery.bridgesmathart.org/exhibitions/2023-joint-mathematics-meetings/realneel

[4] N. Shrestha. "Elevated Rhombicuboctahedron." *Bridges Halifax 2023 Art Exhibition Catalog*, https://gallery.bridgesmathart.org/exhibitions/2023-bridges-conference/neel-shrestha

[5] R. Roelofs. "Elevations and Stellations." *Bridges Conference Proceedings*, Seoul, Korea, 14-19 August, 2014. https://archive.bridgesmathart.org/2014/bridges2014-235.pdf