Space Filling Smith Truchet Variations

H. A. Verrill

Warwick University, UK; H.A.Verrill@warwick.ac.uk

Abstract

This paper describes a method of continuously deforming a Truchet tiling to obtain fractal dragon curves. Since the process is continuous, it can be applied at a varying level across an image, and so used as an image filter. A novel point is the use of hinged tilings in the creation of fractals.

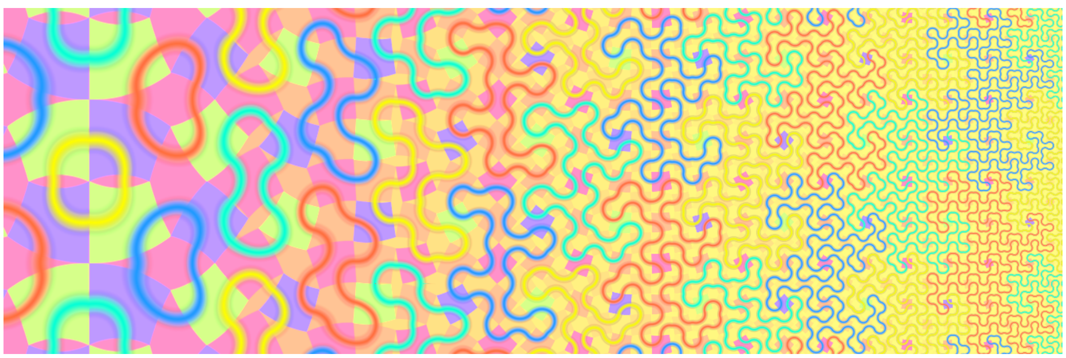

Figure 1: *Continuously varying iteration of a Truchet tiling by hinging the tiles.*

Brief Recap on Space Filling Curves

In 1890 Peano devised a paradoxically "space filling" curve [13]. See [3] and [7] for theory and examples. The closely related Hilbert space filling curve is produced as illustrated in Figure 2, which shows steps from $t = 0$ to 4 of an infinitely applied iterative procedure. There are many related replacement rules for obtaining space filling curves [4]. Art based on such curves can be found for example in [7]. This paper focuses on

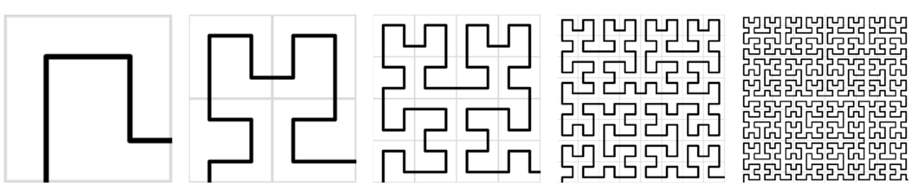

Figure 2: *Derivation of the Hilbert space filling curve, first few steps of an infinite progression.*

a hinged tiling method applied to Smith Truchet tilings, which turns out give rise to fractal dragon curves. Many papers explore other artistic applications of Truchet tilings, e.g., [2].

Continuous Variation of the Replacement Rule

The curves in Figure 2 can be interpolated by a continuous procedure as in Figure 3. Figure 3 (a) shows curves with $t = 0, 0.25, 0.5, 0.75, 1$ from a deformation from $t = 0$ to $t = 1$. In (b), (c), and (d), t depends on the coordinate $(x, y) \in [0,1]^2$, with $t = (\lfloor 4x \rfloor + \lfloor 4y \rfloor)/2$, $(f(4x) + f(4y))/2 - 1$, and $2(x + y) - 0.7$ respectively, where $f(z) = \lfloor z \rfloor + \min(1, 3\{z\})$. The background is coloured according to the value of t, using the colour scale below these images. E.g., red means $t = 1.5$. In (e) t is determined by Perlin Noise.

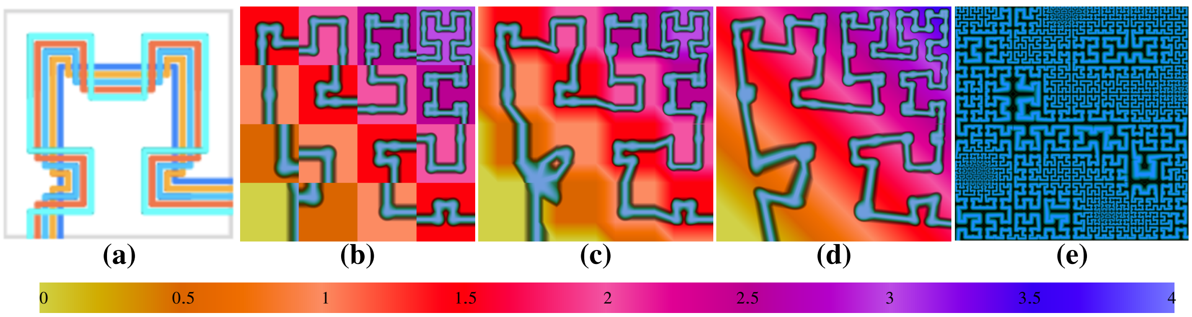

Figure 3: *Variation of level of iteration of Hilbert's space filling curve.*

Heighway's Fractal Dragon

In 1967 John Heighway discovered the fractal dragon space filling curves [6, 14], obtained by repeatedly folding a strip of paper in half, then opening the folds to 90°. It can be obtained by an edge replacement rule, replacing a line segement with an L shape. It is given by an Lindenmayer system, where a string of Ls (left) and Rs (right) describe the direction the path takes. A replacement rule inserts alternately L and R between each symbol to obtain a new sequence [6]. These methods are shown in Figure 4 (a), (b), (c) respectively.

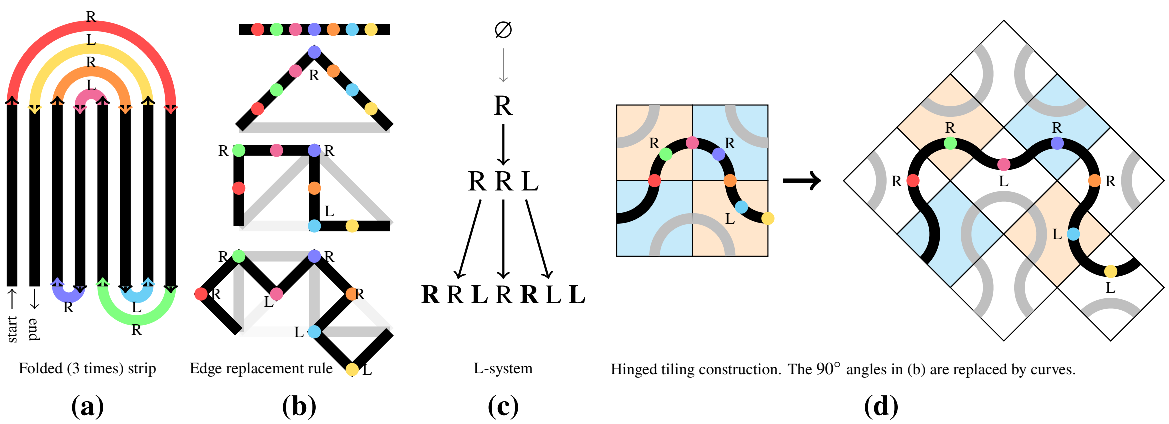

Figure 4: *Construction of Heighway's Fractal Dragon.*

Space Filling Truchet Curves

A Truchet tiling is a tiling by a repetition of a single, square tile. Cyril Stanley Smith [5] popularized Truchet tilings with a now ubiquitous variant that consists of two quarter circles centred at two opposite corners of a square, as in the tiles in the top row of Figure 5. Although the Smith Truchet tiling has been used as a space

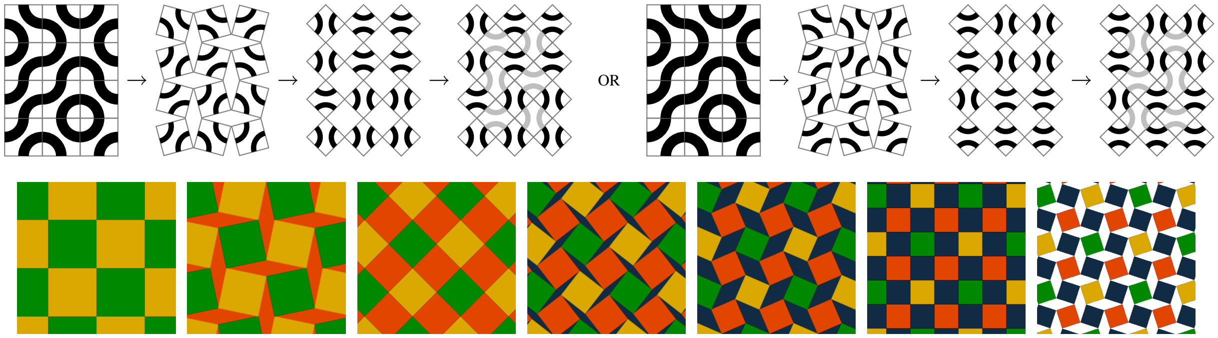

Figure 5: *Hinged Truchet tiling. The top row shows two possible directions of hinging the tiling, giving two operations. The second row illustrates repeated application of the hinging operation.*

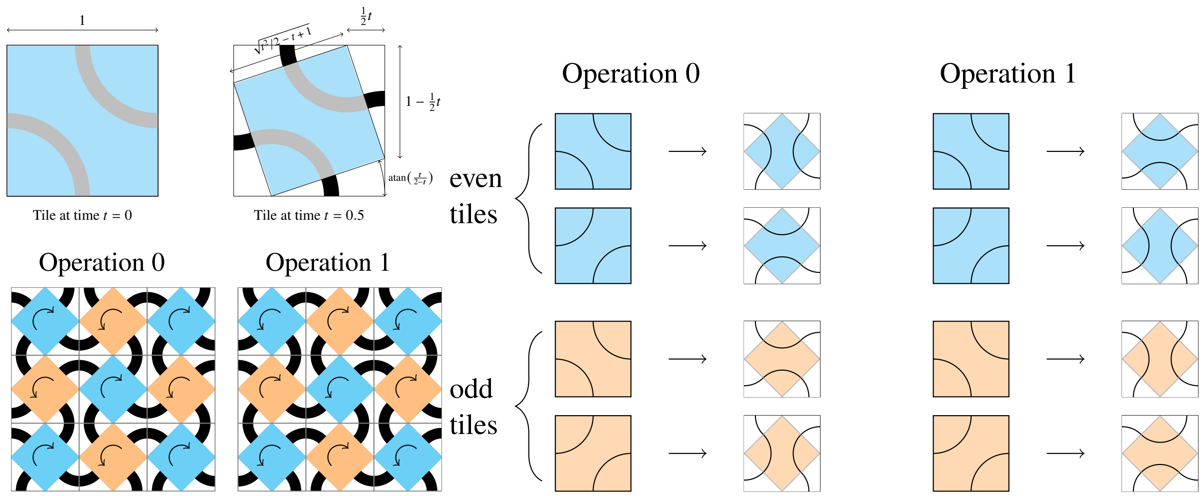

Figure 6: *How new tiles are added to the hinged tiling, for either of the two operations, and how to view the transformations as a replacement rule. Even and odd refer to the parity of the sum of the integer valued grid coordinates of the tile. These are coloured alternately blue and orange on the left.*

filling curve [1], there does not appear to be a space filling iterative tile replacement procedure previously explicitly published in the literature, although the depiction of the fractal dragon curve [14] clearly shows a Truchet type structure. The solution proposed here is to replace the initial tile with two copies, scaled by a factor of $1/\sqrt{2}$, and rotated by 45°, one placed centrally, and one being cut into four parts, forming the corners of the new tile. This is achieved by a hinged tiling method, as in Figure 5. When fully opened, another set of tiles is inserted with alternate orientations, to maintain the original connectivity. Figure 6 shows how alternate tiles have mirrored replacement rules. The repeated application of the replacement rules leads to the fractal dragon curves, as indicated in Figure 4 (d). From the two operations, each infinite binary sequence determines a different fractal dragon curve, as explained in [6], [10], and [9].

The iterative procedure can be continuously varied across an image, as in [7], which uses the Hilbert curve rather than the dragon curve. In Figure 7(a) the iteration level t increases with the distance from the left. In (b) t increases with the distance from the centre. In (c) and (e) t decreases with the distance from the centre. In (d) the pink, green, and blue curves have a constant level of iteration, 0, 1 and 5 respectively. Figure 1 is obtained by starting with the tiles having alternating orientation. The iteration stage t increases from left to right, producing a progression towards the fractal dragon curve.

Summary and Conclusions

This paper illustrates how Smith Truchet tilings are related to fractal dragon curves. The hinged tiling method has the potential to be applied to other curves drawn on hinged tilings. There are a number of mathematical puzzles involved in the computation and artistic depiction of these pictures. For example, colouring problems of the underlying tiling, as well as of the curves. Further examples can be found at [8, 11, 12].

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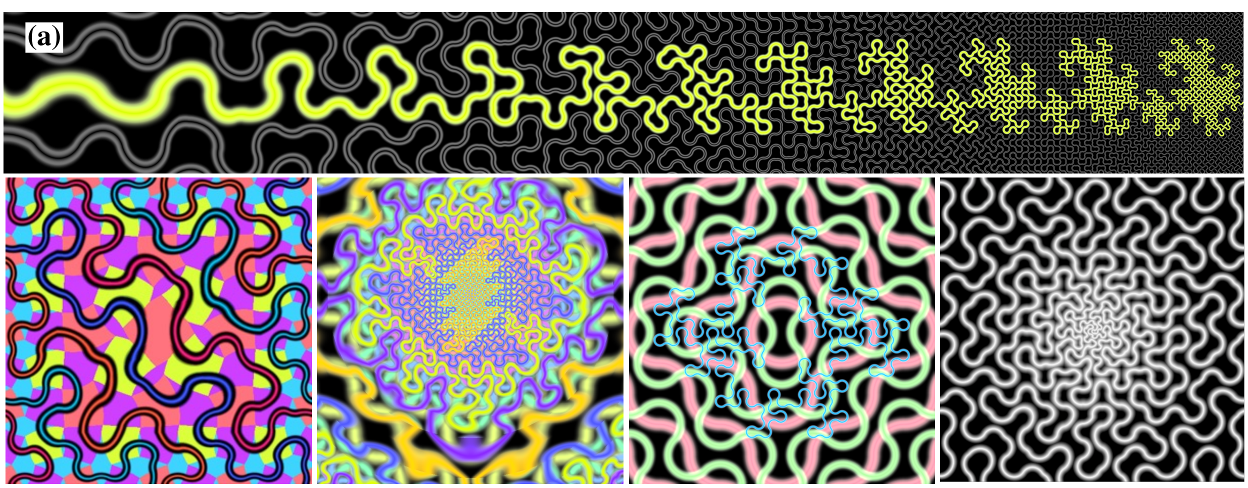

(b) **(c)** **(d)** **(e)**

Figure 7: *Varying hinged tiling fractal Truchet patterns.*