Surface-Tiling Curves

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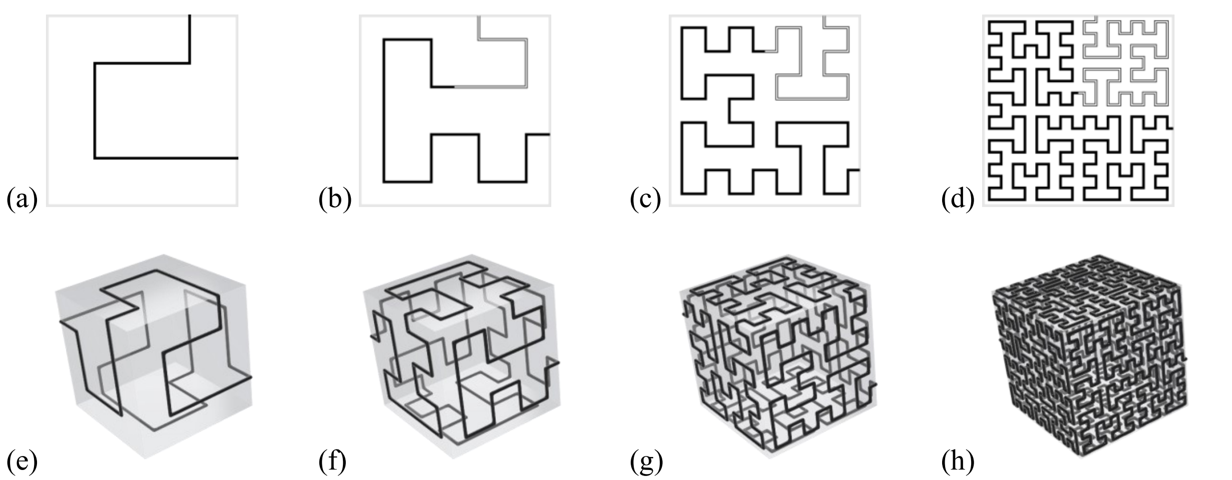
Abstract

This paper explains the process for constructing a space-filling curve, segments of which tile the faces of a regular polyhedron; describes its application to a polyhedron net; and illustrates some of its interesting properties with physical examples.

Introduction

There is a method for constructing a space-filling curve which positions its termini at points allowing copies of the same curve to be connected end to end and applied to the net of a regular polyhedron. The resulting polyhedral shell has faces tiled with segments of a unicursal path: a *surface-tiling curve*.

Each face of a polyhedron includes a space-filling curve which is one segment of the entire curve, referred to here as a tile. Figure 1(a–d) shows tiles for the first four iterations. Rotated and reflected tiles connect across dihedral angles of polyhedrons, wrapping the surface in a continuous loop and comprising the entire curve, as shown in Figure 1(e–h).



(a) (b) (c) (d)

(e) (f) (g) (h)

Figure 1: *Iterations 1–4: (a–d) curve segment on a face (tile), (e–f) entire curve wrapping around a shell.*

Constructing a Curve Tile

Curve *tiles* follow a basic pattern of fractal curve construction: each iteration is made up of transformed copies of the previous iteration, indicated as outlined paths in Figure 1(b–d). Each iteration also includes an adjusted version of the previous iteration (described below).

The construction method alternates for every other iteration, indicated in Figure 2 as A and B. Both methods consist of four steps, each indicated by thin lines with arrows, which place transformed copies of the previous iteration in order in quadrants I–IV. The two methods differ in the third step (↻ 90° vs GLIDE). Construction for each iteration ends with a transformation (indicated by an outlined arrow) of the new tile for its orientation in quadrant I as the basic component of the following iteration. The two methods also differ here (FLIP ↔ vs ↺ 90°).

The basic component for each iteration can be seen as having two parts. One of these parts, indicated by a gray background in Figure 2, is reflected in the last step of each iteration's construction, and becomes a symmetrical version of the basic component. This last step creates the overall path direction necessary to apply the curve to a net.

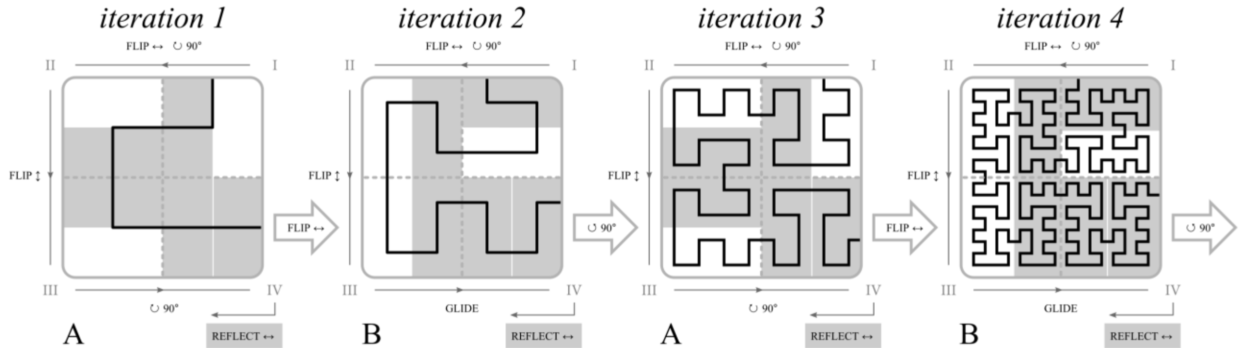


Figure 2: *Alternating construction pattern for a surface-tiling curve tile.*

Applying a Curve to a Net

A surface-tiling curve tile's overall path direction allows an unbroken end-to-end path to be applied to a net by using alternating transformed tiles. Because the tile's path does not branch, only non-branching nets will work. Figure 3 shows the transformation pattern for a tile applied to such a net.

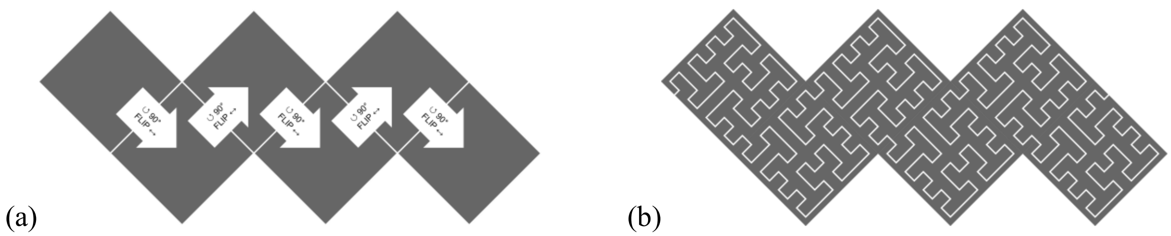


Figure 3: *(a) an acceptable net for a surface-tiling curve with tile transformations indicated, (b) attached tiles applied to the net, creating the whole curve.*

When the net with the applied tiles (the whole curve) is folded into a polyhedral shell (Figure 4), the path traverses all the faces and connects with itself, becoming continuous.

Figure 4: *Several iterations of a surface-tiling curve, each applied to a polyhedral shell (paint on oxidized steel).*

Additional Properties of Surface-Tiling Curves

Stacking

Cube shells can be arranged into stacks that preserve the unicursality of the exterior, visible curve. Figure 4 shows a few of the many possible arrangements.

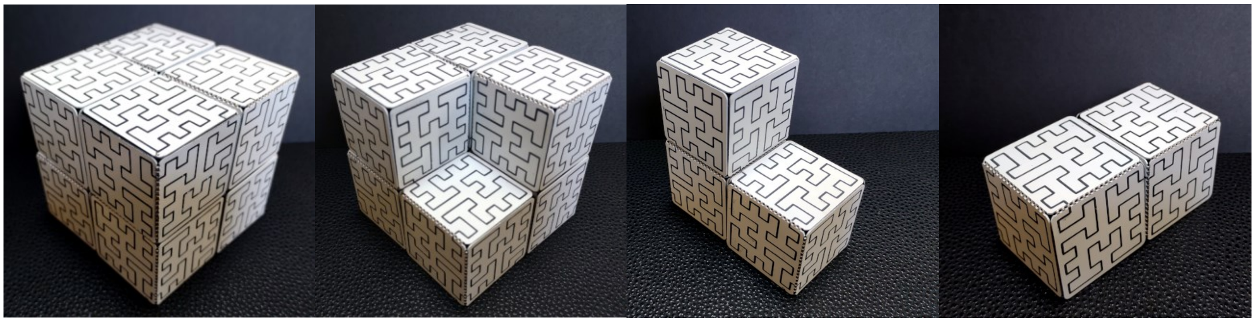


Figure 4: *Various arrangements of stacked shells with surface-tiling curves (paint on steel)*

Bending / Twisting

Even numbers of tiles arranged in the basic net pattern used in Figure 3 can be bent into a continuous ring. Odd numbers of tiles can be bent and twisted into a Listing band (a.k.a. Möbius strip), shown in Figure 5. Both arrangements maintain unicursality.

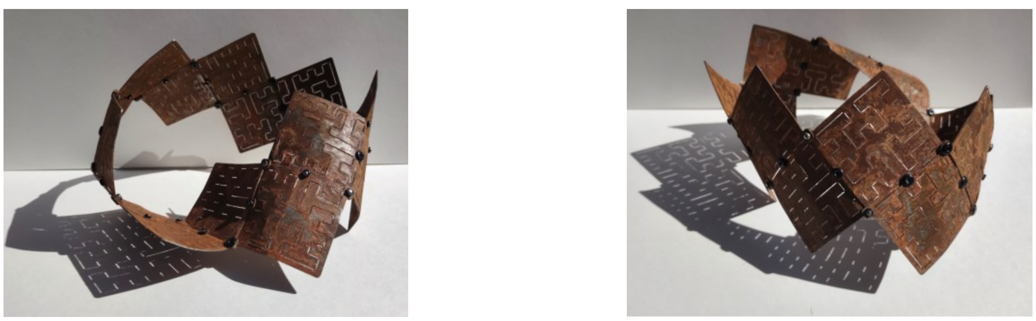


Figure 5: *Different views of a surface-tiling curve on a band (waxed oxidized steel).*

Slicing

When a shell is cut along the path of a surface-tiling curve, the shell is sliced into two mirrored pieces with mirrored, branching nets, shown in Figure 6. The two pieces are not locked in place and can be separated with two slide moves.

(a) (b) (c)

Figure 6: *a) shell cut along the path of a surface-tiling curve,*
b) unfolded net after slicing, C) separated shell pieces (paint on steel).

Clarifications / Generalizations

Platonic polyhedra were chosen for the exploration in this paper for two reasons: 1) each is composed of identical faces and 2) the shapes of those faces are easily mapped into Gaussian and Eisenstein domains, keeping surface-tiling curve construction simpler and more consistent with other curve construction methods [3].

In the examples given, the term "polyhedron" is used although all the examples above are cubes. Surface-tiling curves also exist for the triangle-faced regular polyhedra. These three polyhedra can use the same tile, iterations shown in Figure 7.

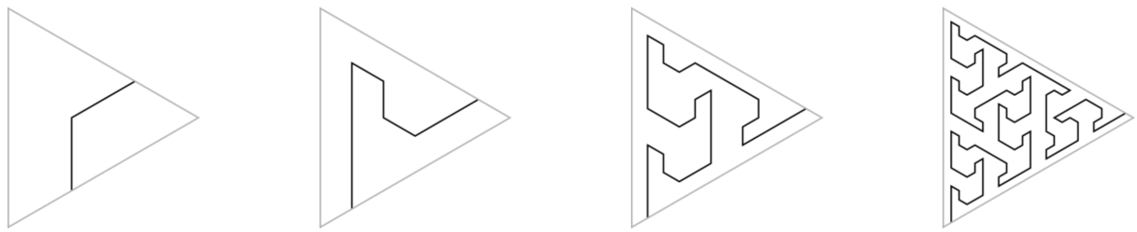
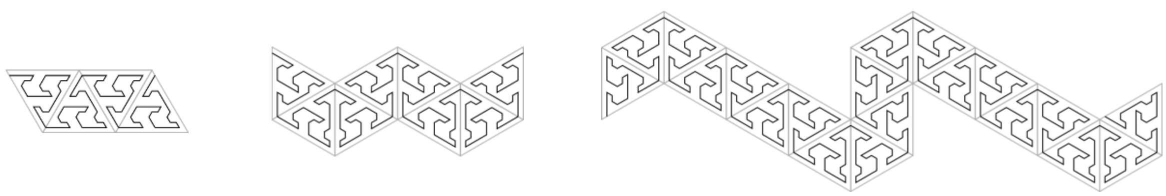


Figure 7: *Iterations of a triangle-based tile.*

Non-branching nets of all three triangle-faced regular polyhedra can encompass surface-tiling curves (Figure 8). Bending / twisting nets and slicing shells work the same as for cubes.



(a) (b) (c)

Figure 8: *Surface-tiling nets: a) tetrahedron, b) octahedron, c) icosahedron.*

There is a similarity between surface-tiling curve tiles and the Hilbert curve. However, the Hilbert curve cannot tile the faces of a polyhedron without adjustment. Its iterations alternate between entering / exiting a square containing the curve on the same edge and on opposite edges [2]. It is the feature of entering / exiting adjacent sides of containing shapes which allows the polyhedral face tiling of surface-tiling curve tiles for polyhedrons with square and triangle-faced polyhedra.

The collection would be complete with a surface-tiling curve for the dodecahedron. However, a continuous linear path with identical tiles on each face is not possible with a dodecahedron net because the pentagon requires two different general path directions across connecting faces: a path entering one side would need to exit an adjacent side sometimes and an opposite side other times (diagram in supplementary document). A look at the 340 non-branching dodecahedral nets shows that not one is composed of purely adjacent-side-exiting or opposite-side-exiting arrangements [1].

A curve with tiles spanning more than one face (such as a curve segment contained in an octagon made up of two connected pentagons) might address this issue and is an avenue of future investigation.

References

[1] E. Pegg Jr. Wolfram Demonstrations Project. *Path Nets for Dodecahedron and Icosahedron*. 2018. https://demonstrations.wolfram.com/PathNetsForDodecahedronAndIcosahedron/

[2] H. Sagan. *Space-Filling Curves*. Springer-Verlag, 1994.

[3] J. Ventrella. *The Family Tree of Fractal Curves*. Eyebrain Books, 2019.