The Oloid and the Evertible Cube: 3D Design and Printing

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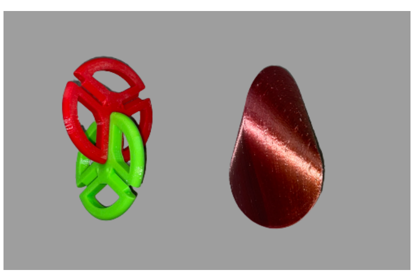
Abstract

The oloid was discovered by Paul Schatz in the 1930s while investigating the geometry and kinematics of the evertible cube. Using 3D design and printing technologies, we seek to make sense of the generative mathematics and art behind Schatz's work and further bring engaging artifacts and learning activities to school teachers and children.

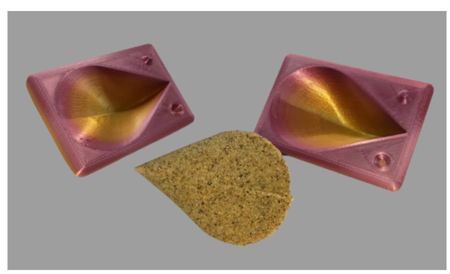
Introduction to the Oloid

The oloid, discovered by Paul Schatz in the early 1930s [4, 5], is a jewel of mathematical art, kinetically inviting and mathematically generative. It can be readily modeled to the general audience using paper templates in multiple styles [1]. The oloid can also be developed from two interlocking circles that have the same radius and are orthogonal to each other, with one going through the center of the other. The circles can be cut out of cardboard or, as shown in Figure 1a, 3D printed for more accuracy and visual appeal.

In working with school children, I designed some grooves around a circular disk with a locking key so that two copies of the circle snap into each other with a click (Figure 1a). The pair can be rolled on a flat surface to map out the oloid surface. From the two interlocking circles, we can create the oloid as a convex hull of the pair. To allow for the use of Kinetic® or sticky sand, I further designed an oloid casting mold that can quickly turn a handful of sticky sand into an oloid (Figure 1b). These models provide a meaningful and hands-on introduction to the intriguing "form-feeling" around the oloid. The curious minds can continue to delve into the world of mathematics using [2] as a reference.



(a)

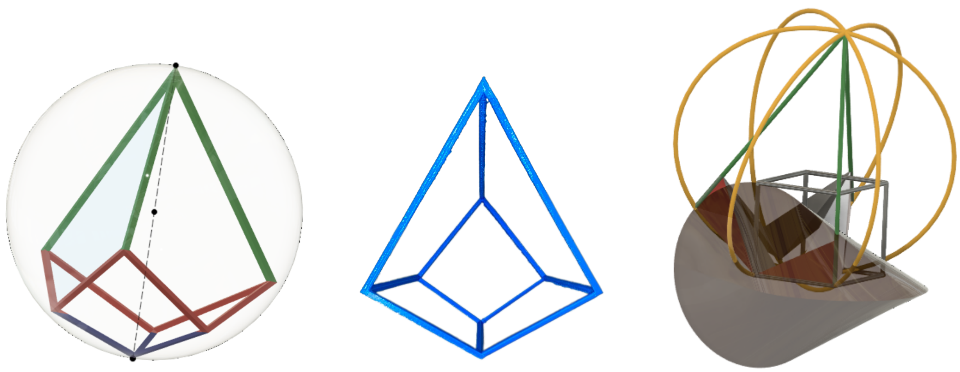


(b)

Figure 1: *3D printed oloid models: (1) interlocking circles and oloid, (2) molds and sticky-sand oloid.*

The Evertible Cube

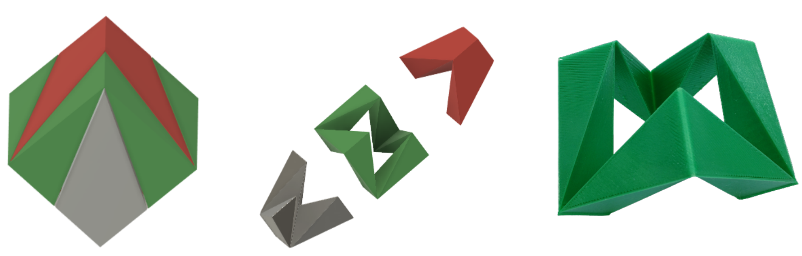
Using two circles, we can approach the oloid as a kinetic experiment and further create a 3D printable model using design technologies. Schatz, however, discovered the oloid while investigating the eversion of the cube, where the cube is cleverly transformed into a right-angled hexahedron, also known as a deltohedron [4, 6]. In the process of eversion, it is as if the cube was stretched at one end and pressed at the other, along one major diagonal, so that each of its square faces becomes a deltoid or kite-like shape [4, p. 25]. All the polar edges of the cube remain perpendicular to the central edges during eversion, as illustrated in Figure 2a. The resultant deltohedron was initially elusive during my reading of [4] until I designed a 3D printable model as shown in Figure 2b. It is difficult to show the three-dimensional structure in a 2D image, but the real object is illuminating, once we hold it in our hands. Essentially, there are two opposing trihedra connected by six central edges of the cube while maintaining the orthogonal relationship between all polar edges and their corresponding central edges (Figure 2a). Most interestingly, all the eight vertices of the deltohedron are on the same sphere (see Figure 2a), which can be readily observed when attention is drawn to the many orthogonal relationships [4, p. 26]. The elongated major diagonal of the cube serves as the diameter of the circumsphere. Figure 2c shows how the deltohedron is connected to other structures to be described.



(a) **(b)** **(c)**

Figure 2: *The deltohedron: (a) a 3D designed deltohedron with its circumsphere and two polar tridehra (in green and blue), and the central belt (in red), (b) a 3D printed deltohedron, (c) a deltohedron and its connections to the cube, the evertible cube, and the oloid.*

Once we come to appreciate the properties of the deltohedron, we can construct the evertible cube using Schatz's approach. As illustrated in Figure 3a, a ring-like cube belt of six tetrahedra are constructed along the central edges of the cube. Each tetrahedron is made up of four right-triangle faces. The shortest edge of each tetrahedron, which is part of the polar edge, joins the six solids in a ring, with a theoretical length between zero and $\frac{\sqrt{3}}{3}$ of the cube edge length. When it takes the maximum value of $\frac{\sqrt{3}}{3}$ of the cube edge, the six-piece ring will flatten twice to an equilateral triangle with no gap during its eversion (cf. Figure 5b).

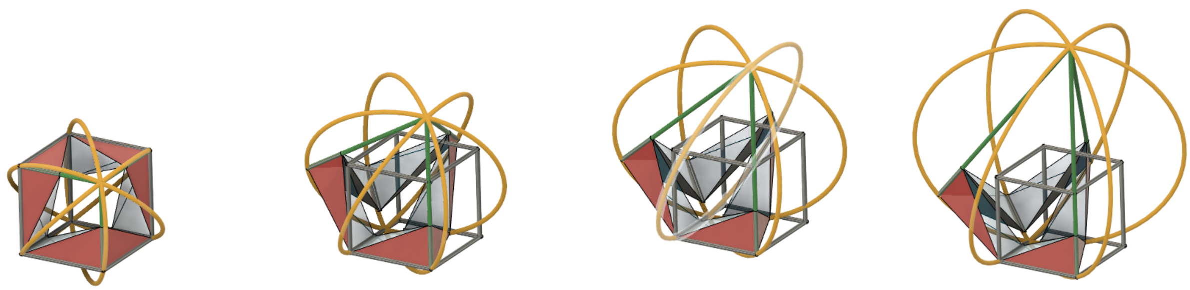


(a) **(b)** **(c)**

Figure 3: *The evertible cube: (a) 3D design, (b) 3D design pulled apart, (c) the evertible cube ring 3D printed with flexible filament.*

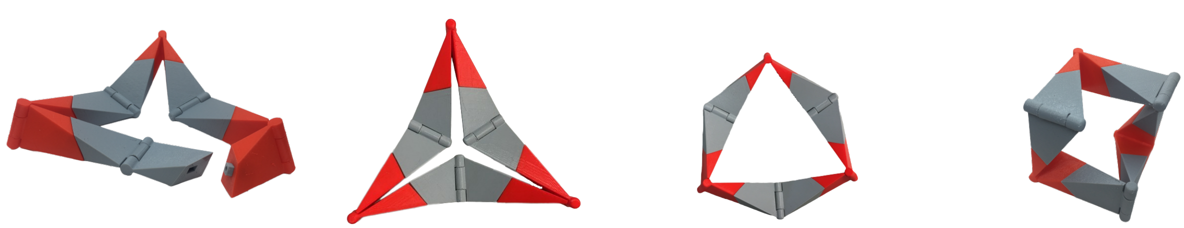
In my designs, I set the polar edge of the evertible cube to one-half of the cube edge so that there is some gap in the middle (see Figure 5b), which allows smooth physical manipulation. The cube ring divides the cube into three parts, including two identical polar corners. The cube ring, with some joint reinforcement, can be 3D printed using flexible filament (Figure 3c), for physical eversion. As the cube ring goes through its eversion cycle, the six short polar edges, when extended, form an expanding/contracting deltohedron (Figure 4). The deltohedron starts from a cube, reaches infinity four times, and contracts back to its reference cube four times [4, p. 27]. Meanwhile, the outside of the cube becomes the inside of the cube, hence its name – the evertible cube. Again, this remarkable cycle is elusive on paper. I thus designed several 3D printable models to illustrate this rhythmic process at different moments of the cube eversion.

The six-piece cube ring is the gem of Schatz's cube eversion, and is often called the evertible cube because of its generative nature. While it can be 3D printed using flexible filament, I was curious about the possibility of a hinged version for in-depth explorations. Indeed, hinges can be utilized to join the six tetrahedra into each other. The whole design can be 3D printed in one piece with proper support. Upon close examination, however, I realized that the cube ring could be sliced into six identical hinged pairs using a plane that contains the midpoints of the central edges of a cube, namely a hexagonal section of the cube. We now can print six copies of the same hinged joint and, taking advantage of their intrinsic symmetries, assemble them into a cube ring – an evertible cube. I further added some small pins to facilitate their alignment (see Figure 5). The hinged evertible cube provides a smooth and playful hands-on introduction to the fascinating ideas behind Schatz's text.



(a) (b) (c) (d)

Figure 4: *During the cube eversion, the circumsphere of the deltohedron expands and contracts.*

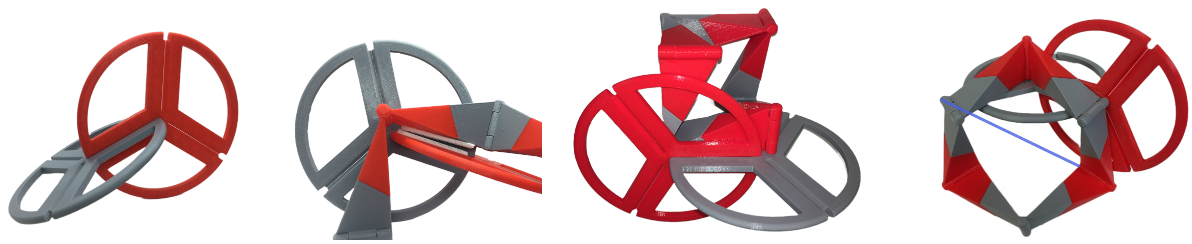


(a) (b) (c) (d)

Figure 5: *Six hinged joints can be assembled for an evertible cube.*

Toward a Kinetic Synergy

Now we have the interlocking circles, the oloid, and the evertible cube as well as the cube in the background of all our discussions. How do they all work with one another? When I mounted the hinged cube ring, using some velcro tape, on the two large interlocking circles (Figure 6a), I saw the connections among all the components. It is worth noting that, in my designs, the cube edge length is slightly longer than the circle radius to allow the ring to move freely around the circles. Theoretically, they should be of the same length. Further, as shown in Figure 6b, the short polar edges of the fixed tetrahedron are positioned at the circle centers and are orthogonal to the two circles, respectively. The blue segment in Figure 6d is a cube major diagonal, which remains constant during the eversion while enveloping the oloid (see [3] for a virtual simulation). It is a very comforting cycle of expansion and contraction, and, as Schatz suggested, one should "close[s] one's eyes and follow[s] the movement with the breath, as it were" [4, p. 29]. Although it is visually impossible to see the overlapping structures in a 3D printout containing the resultant oloid with the original cube, the evertible cube ring, and the circumsphere, it is possible to design such a comprehensive 3D model in order to savor the beauty of Schatz's ingenuity in connecting geometry to kinematics and art. Figure 2c summarizes all the major ideas with the yellow circles depicting the circumsphere of the deltohedron.



(a) **(b)** **(c)** **(d)**

Figure 6: *Mounting the evertible cube on the interlocking circles: (a) two interlocking circles, (b) mounting the evertible cube on the circles using adhesive or velcro tape , (c) the evertible cube at its original cube position, (d) the evertible cube cycles around the circles.*

Summary and Conclusions

There is a world of treasure in Schatz's work [4]. Although the text is sometimes elusive due to the natural complexity of the endeavor and the limitations of 2D images, we can make sense of his ingenious and generative ideas using 3D design and printing. This is an open invitation to all — Take an excursion in the Schatz world and make your (re-)discoveries!

Acknowledgements

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