Polar Zonohedral Helices and Clusters

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Abstract

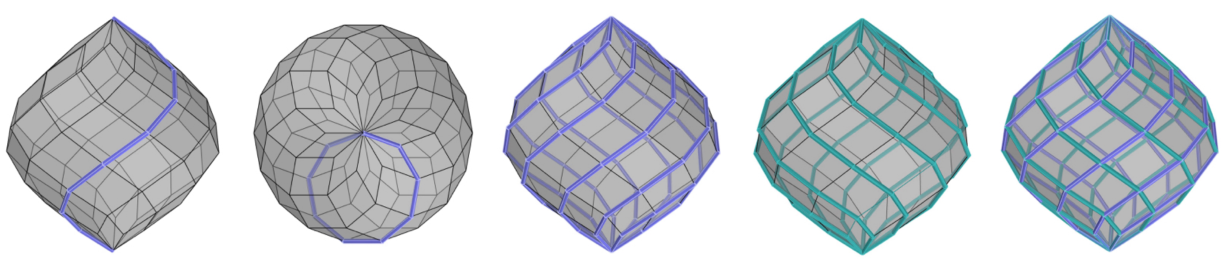
Polar zonohedra (PZ) are a class of polyhedra whose rhombic faces lend themselves well to decoration. In this paper, I discuss two new explorations that focus more on the 1- and 3-dimensional aspects of the shapes, namely, their edges and their complete volumes. In the first exploration, I create forms extruded from the helix-like shapes embedded in the edges of PZ. In the second, I examine and enumerate the possibilities for clusters of PZ that fill 3-dimensional space locally around a single point. Both avenues yield numerous possibilities for artistic work.

Introduction to Polar Zonohedra

Polar zonohedra (PZ) are an especially beautiful, rotationally symmetric class of polyhedra all of whose faces are rhombi with the same edge lengths [1][6]. It is easiest to picture a PZ as emanating from the origin with an n-fold axis of symmetry in the z direction. Each $PZ(n,\theta)$ is defined entirely by n, the number of equal length generating vectors (spaced at angles of $(360/n)°$ around the z axis), and θ, the "pitch angle" of those vectors relative to the xy plane. When θ is close to 0°, the PZ will have a very flat, pancake-like shape; when θ is close to 90°, it will have a very pointy, cigar-like shape; and in between it will have a shape reminiscent of a football. Hart [1] enumerates many other interesting properties of this class of shapes. In a previous Bridges paper [8] I explored surface decoration of PZ using Islamic geometric patterns. Here I explore two new ways of deriving artistic work from these compelling shapes.

Exploration 1: Polar Zonohedral Helices

One of several compelling visual aspects of PZ is that their edges comprise two symmetric sets of paths running from pole to pole in opposite directions, each of whose vertices lie along a helix (see Figure 1).



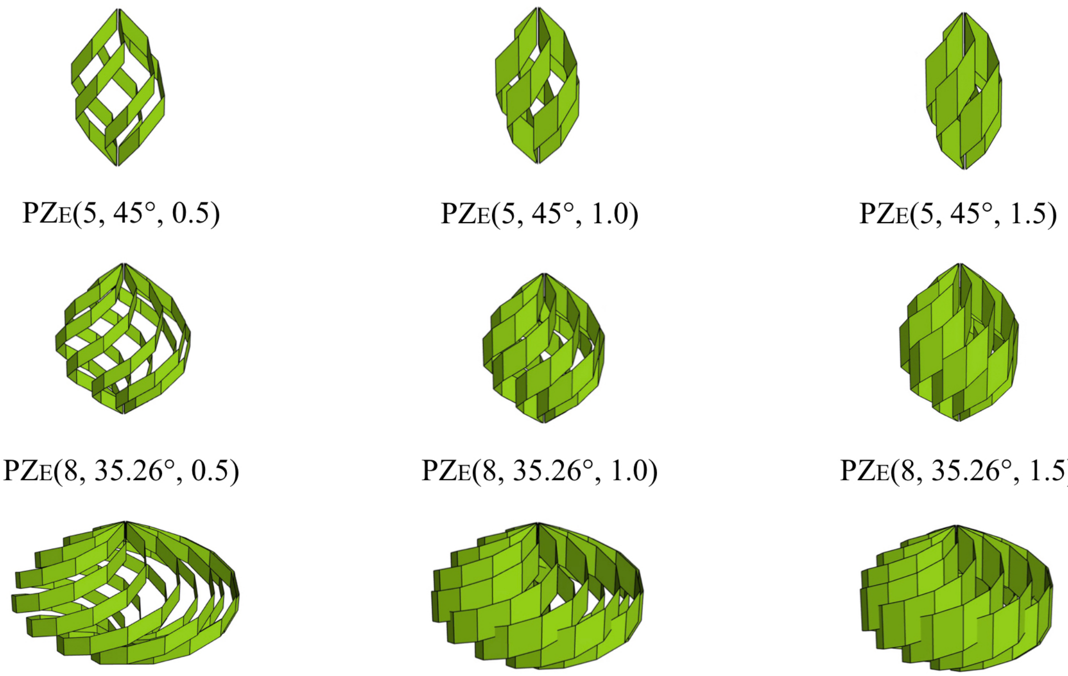
(a) (b) (c) (d) (e)

Figure 1: *Edge helices on a 9-fold PZ. A single helix from the front (a) and top (b); all CW helices (c) and all CCW helices (d); all helices (e) with every edge used, polar edges used twice.*

Henceforth I will use the term "helices" loosely to refer to these sets of edges, though they might more accurately be called "discretized helices" since each set forms a piecewise linear curve and not a smooth continuous one. When an n-fold PZ is viewed along its axis, its edges group into n clockwise and n counterclockwise helices. Each helix comprises n edges whose projection on a plane perpendicular to the axis is a regular n-fold polygon. Each edge is part of exactly one such helix, except for the edges at the poles, each of which is part of one helix in each direction.

Extruding the Helices

While considering ways in which to emphasize this especially beautiful aspect of PZ, it occurred to me to focus on one of the two sets of helices (either the CW or CCW helices), and to try extruding them along the direction of the polar axis (i.e. vertically, assuming the PZ is oriented with its axis in the z direction). This immediately yielded some beautiful and compelling forms (see Figure 2).



PZE(5, 45°, 0.5) PZE(5, 45°, 1.0) PZE(5, 45°, 1.5)

PZE(8, 35.26°, 0.5) PZE(8, 35.26°, 1.0) PZE(8, 35.26°, 1.5)

PZE(11, 20°, 0.5) PZE(11, 20°, 1.0) PZE(11, 20°, 1.5)

Figure 2: *Examples of extruded PZ helices $PZ_E(n, \theta, d)$ for various values.*

In addition to the existing parameters for the base PZ (n and θ) this effectively adds a third parameter d (for distance or depth), so that we can notate such shapes as $PZ_E(n, \theta, d)$ where the subscript E denotes extrusion. Expressing d as a multiple of the PZ edge length makes it independent of the shape's overall scale. Thus, a value of $d = 1$ means that the extruded faces are rhombi, with values of $d < 1$ and $d > 1$ yielding flatter and taller parallelograms, respectively. Technically we could include a fourth parameter for CW or CCW handedness, but since the overall symmetry of the base PZ makes this difference equivalent to a simple mirror reflection, I have opted to omit it for clarity. As Figure 2 demonstrates, the new parameter d has a dramatic effect on the visual appearance. For large values of n (e.g. $n = 11$, Figure 2, bottom row), a rather small value for d is required to avoid collision of the extruded helices. Conversely, smaller values of n (e.g. $n = 5$, Figure 2, top row) allow for larger values of d.

Works Derived from Extruded PZ Helices

I selected an example I found particularly pleasing—$PZ_E(8, 35.26°, 1.2)$, slightly more extruded than the central shape in Figure 2—and used it as the basis for creating the first works in a series I call *Galaxy Lamps* because of their spiral arms (see Figure 3). Obviously, many more possibilities exist to be explored.

(a) (b) (c) (d)

Figure 3: *Three of the author's Galaxy Lamps. Top view shown in (c).*

Exploration 2: Polar Zonohedral Clusters

There is much existing work about how general zonohedra (not necessarily polar) can pack together in space [3][4][5][7], which they do quite naturally given their derivation from a fundamental set of generating vectors. The only PZ that can pack 3-dimensional space completely by themselves are extremely special cases such as the cube, rhombohedron, and rhombic dodecahedron. Their space-filling properties rely on additional symmetries beyond the symmetries granted by their generation via a set of originating PZ vectors.

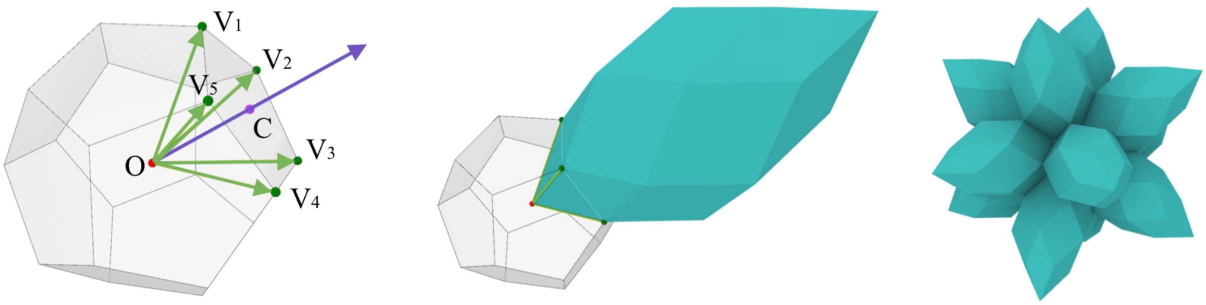
I instead became fascinated with a different question: how many ways can PZ be arranged so as to completely fill space locally around a **single** point in 3-dimensional space? Many years ago I created a sculpture (*12-pointed Islamic Star*, see Figure 7) whose external shape comprises exactly such a cluster. (This cluster also appears in work by Kabai [3] and Towle [7].) Given how beautifully that sculpture turned out, and once I realized that it could be viewed as a cluster of 12 identical 5-fold PZ arranged in dodecahedral symmetry, I was inspired to look for more such clusters.

Conceptualizing Polar Zonohedral Clusters

Using my original sculpture as a guide, I realized that it could be generated from a regular dodecahedron in the following way (see Figure 4):

- Take as the originating pole the exact center of the dodecahedron (Figure 4(a) - point O in red).
- Take as the set of generating vectors the set of vectors that emanate from O to the vertices of one pentagonal face (Figure 4(a) - points V_i in dark green, vectors $\overrightarrow{OV_i}$ in light green).
- Take as the polar axis the vector emanating from O through the face center C (Figure 4(a) - center point C in pink, vector $\overrightarrow{OC}$ in purple).
- Use these to generate a 5-fold PZ emanating from the center along the polar axis (Figure 4(b)).
- Repeat this process for all 12 faces of the dodecahedron (Figure 4(c)).

Analyzing this example helps to conceptualize, and then enumerate, all possibilities for clusters of PZ surrounding a point. If we consider a given point O and a set of points V_i with centroid C, we can see that the only way the vectors $\overrightarrow{OV_i}$ can serve as generating vectors for a PZ is if the points V_i form a regular n-fold polygon lying in a plane perpendicular to axis $\overrightarrow{OC}$, since this is the only possible configuration in which all the vectors $\overrightarrow{OV_i}$ will be of equal length and distributed evenly around axis $\overrightarrow{OC}$. Additionally, the length of $\overrightarrow{OC}$



(a) (b) (c)

Figure 4: *Generation of a PZ cluster from a regular dodecahedron.*

must be non-zero; if O and C are coincident then the PZ has pitch angle of of 0° and becomes degenerate. Meanwhile, for a set of PZ to completely fill space locally around the point without overlap, each pair of adjacent V_i must belong to exactly two sets of $\overrightarrow{OV_i}$ from two different polygons. This guarantees that adjacent PZ generated from those polygons will share a face and therefore that there will be no "gaps" around the central point. Thus we conclude that the set of points V_i must be the vertices of a polyhedron whose vertices all lie on a circumsphere centered on O, and all of whose faces are regular polygons.

Enumerating Possible Polar Zonohedral Clusters

Armed with this conclusion, it becomes a simple matter to enumerate all of the polyhedra that fulfill this requirement. They belong to 5 well-known families of polyhedra:

1. The Platonic Solids (5 members)
2. The Archimedean Solids (13 members)
3. The Regular Prisms (infinite)
4. The Regular Antiprisms (infinite)
5. The Circumscribable Johnson Solids (25 members, 19 of which are valid for PZ clusters)

The only group on this list which may be slightly unfamiliar is the last one. The Johnson Solids [2] are a well-known and documented class of 92 polyhedra which are convex and have only regular polygons as faces; the "circumscribable" condition simply eliminates all of these whose vertices are not coincident on a single sphere. Further, 6 of the members of this group have faces that are coplanar with the centers of their circumspheres (the "degenerate" condition mentioned above), thus reducing the number of viable PZ clusters in this group to 19. Thus there are 37 distinct PZ clusters, plus the infinite sets of prism- and antiprism-derived PZ clusters (two of which are equivalent to two of the Platonic Solids).

Figure 5 illustrates the various families of polyhedra and their resulting PZ clusters. Individual PZ within clusters are colored by value of n (3-pink, 4-purple, 5-turquoise, 6-dark green, 7-light green, 8-yellow, 10-orange) so that the rotational symmetries are easily spotted.

Some Interesting Properties of Specific PZ Clusters

Figure 6 shows a handful of PZ clusters at larger scale so we can appreciate their full beauty and examine some interesting properties that emerge in certain cases.

Figure 6, row 1 shows that all but one of the Platonic Solids yield PZ clusters whose hulls are themselves recognizable polyhedra (or in one case, a group of them). Green highlights show the mapping of a single

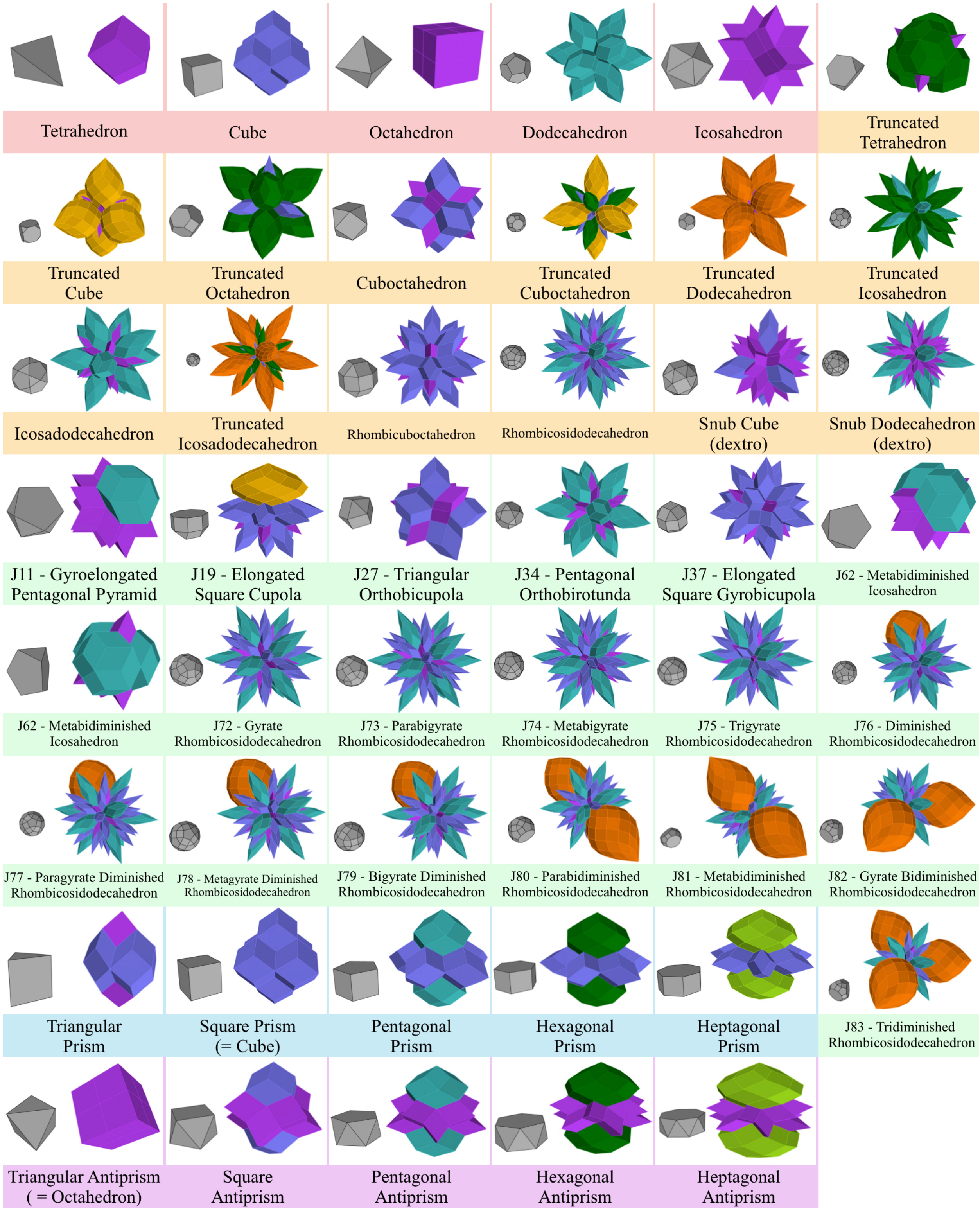


Figure 5: *All valid PZ clusters filling space locally around a point. Only the first 5 from the infinite Regular Prism and Regular Antiprism families are shown. Each PZ cluster is shown as large as possible with its generating polyhedron at the same scale. Label shading indicates originating family (Platonic-red, Archimedean-yellow, Johnson-green, Regular Prism-blue, Regular Antiprism-purple).*

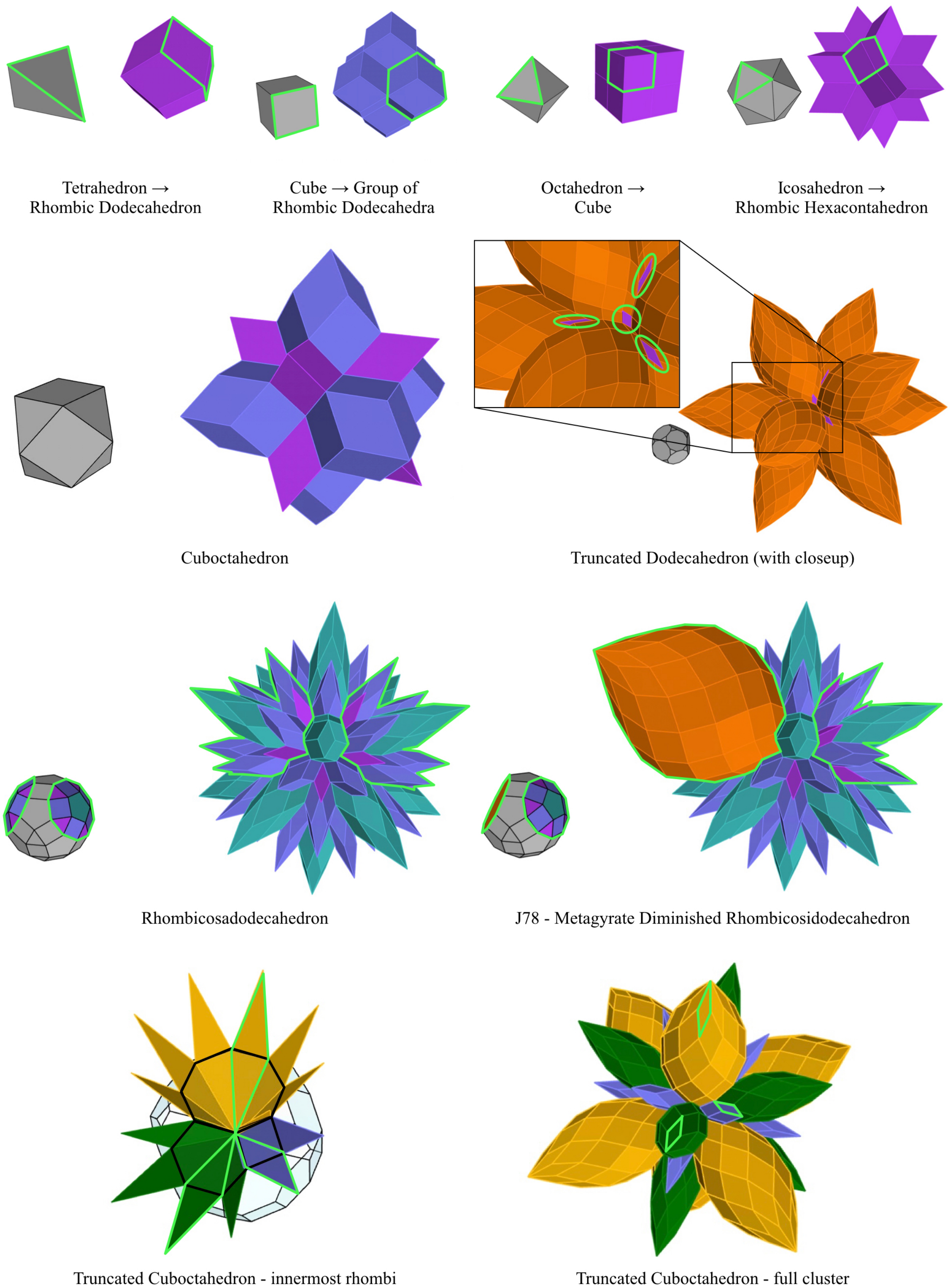


Figure 6: *Examples of PZ Clusters illustrating interesting properties.*

face to its corresponding PZ in the cluster. In the case of the octahedron, each face yields a cube, and the eight cubes have coplanar faces yielding a larger cube as the hull.

Figure 6, row 2 illustrates how variations in polyhedral face sizes dramatically effect the appearance of the resulting PZ cluster. The cuboctahedron, whose square and triangular faces are similar in size, yield an equally balanced PZ cluster. In contrast, the truncated dodecahedron, whose triangular and dodecagonal faces are quite different in size, yields a PZ cluster where the orange 10-fold PZ dwarf the nearly-invisible pink 3-fold PZ (circled in the magnified inset). These examples also show how the Archimedean solids lead to forms that visually appear as compounds, especially when colored by *n* value as is done here.

All of the 19 valid circumscribable Johnson Solids are derived from Platonic or Archimedean Solids through two operations: diminishment and/or gyration. These operations are possible wherever a set of faces of the original polyhedron share a perimeter that is itself a regular polygon. Diminishment refers to removing this set of faces to expose the larger internal regular polygon. Gyration refers to rotating this set of faces around the center of the internal polygon, thereby re-aligning the faces differently with respect to the rest of the polyhedron. These operations have the corresponding effects of replacing several PZ in the originating PZ cluster with a single PZ with large *n*, or rotating a whole group of PZ in place, respectively.

Figure 6, row 3 shows how the Johnson solid J78 is derived from the rhombicosidodecahedron, and the effect this has on their PZ clusters. The precise, if cumbersome, name of J78—Metagyrate Diminished Rhombicosidodecahedron—tells us exactly which operations are at play: one gyration and one diminishment, with the “meta” indicating that these operations occur on internal polygons that are oblique (rather than parallel) to each other. The rhombicosidodecahedron yields so many Johnson solids precisely because it has many occurrences of the same set of 11 faces: a pentagon and its surrounding 5 equilateral triangles and 5 squares, whose perimeter forms a decagon. The green outlines show where the operations occur on the original polyhedron, and which PZ in the cluster are affected. The diminishment occurs on the left side of the shape, replacing the 11 faces with a single decagon. This replaces the original 11 PZ with a single, large 10-fold PZ. The gyration, on the right side of the shape, is more difficult to see immediately. An identical group of 11 PZ is, in this case, simply rotated by 36°. Examination of the outlines show this subtle difference.

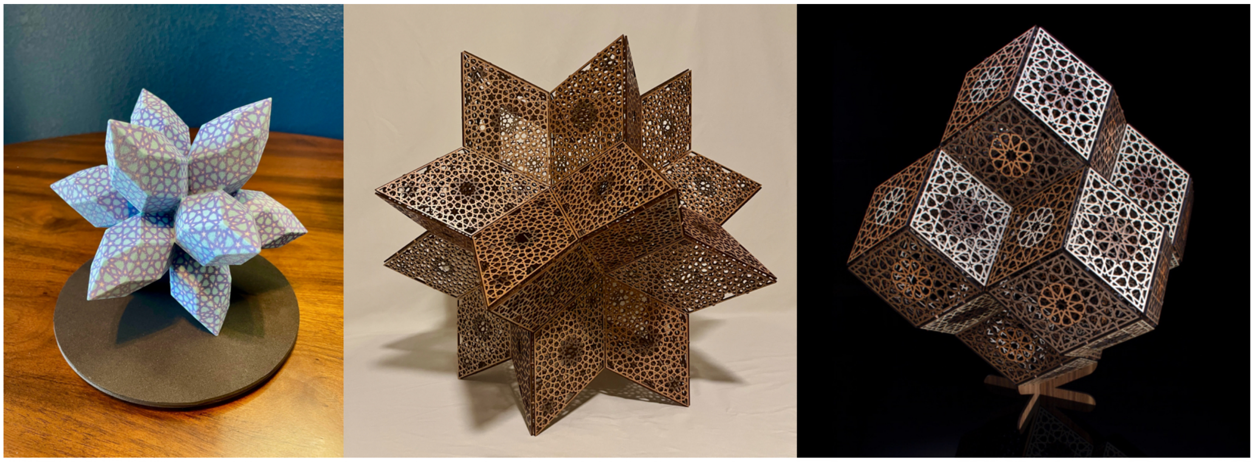
Figure 6, row 4 explains the reason for the initially surprising observation that within each PZ cluster, the rhombi at the outer tips are all identical, regardless of the values for n and θ of their individual PZ. On the left is a (mostly transparent) truncated cuboctahedron with three adjacent faces outlined in bold black lines. From the center we generate the innermost rhombi for these three faces, shown in purple (4-fold), green (6-fold) and yellow (8-fold). We already know that all of the rhombic angles at the tip of each individual PZ are identical by construction. As this figure reminds us, adjacent PZ in a cluster always share a face, therefore, *all* of the pictured rhombi are identical to each other. Since each PZ is symmetric end-to-end, it follows that the corresponding rhombi out at the tips are also identical.

Works Derived from Polar Zonohedral Clusters

In addition to the original sculpture which accidentally implemented the dodecahedral PZ cluster, in the past year I have created two additional sculptures from the PZ clusters generated by the cube and icosahedron (see Figure 7). These particular clusters were chosen because they satisfy the much more stringent criteria that the same Islamic geometric pattern can be applied to all faces [8]. Moving forward I look forward to creating works without such surface pattern constraints using some of the other clusters pictured here.

Conclusion and Future Directions

By considering two aspects of polar zonohedra—their edge helices and their ability to cluster around a single point—I have opened up two fruitful avenues for artistic exploration. I have only scratched the surface of



(a) (b) (c)

Figure 7: *Three of the author's pieces based on PZ Clusters: (a) 12-pointed Islamic Star (generated from a dodecahedron), (b) Jaali Star (from an icosahedron), and (c) Jaali Diamonds 2 (from a cube).*

each one. There are literally an infinite number of possible extruded PZ helices $PZ_E(n, \theta, d)$ waiting to be turned into additional lamps or other sculptural works. And so far I have only implemented works based on three of the PZ clusters enumerated here. I intend to create more works based on these two categories of beautiful shapes, and hope that this paper will inspire others to do the same.

Acknowledgements

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