Graph Theory and Arts: A Course Retrospective

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Abstract

During a course conducted in 2022, the participants investigated connections between mathematics and art. Based on a brief introduction to graph theory and inspired by some existing mathematical artworks, the participants—in this case, high school students—created representations on their own. This paper discusses underlying motivations, the course concept, an approach to idea generation, and the resulting artworks created by the course participants.

Math, Arts, and Teaching

When does one learn about connections between mathematics and art? In art classes in elementary or middle school, students see a basic connection, when they learn how to integrate linear perspective into their sketches. Here, mathematics serves more as a tool rather than as an artistic part of the artwork. Also, it is mostly not explicitly discussed as mathematics. In general, the school curriculum does not aim for an investigation of other connections between mathematics and arts. Mathematics classes tend to focus on calculations and on finding solutions for a given problem instead of considering the playful moment mathematics also provides. Hence, students—especially after high school—are unaware of connections between mathematics and arts.

When it comes to these two subjects, students usually have a strong tendency towards one or the other. Incorporating pieces of mathematical art in the classroom setting piques unexpected interest in the discipline previously perceived as uninteresting. Thus, combining two disciplines—which appear to be completely distinct at first glance—extends the horizon of the students and results in more curious minds and learners. This broadening of horizons is what the course presented in this paper aimed for.

Here, we will discuss a retrospective on a course that combined graph theory and arts. The participants were German high school students, for whom interdisciplinary connections are not part of the curriculum. They were chosen to take part in summer schools for gifted students, for which this course was designed. Other such courses have previously been discussed at Bridges (in [3][5] for example) demonstrating ways of integrating mathematics and art into a single course at various educational levels. We put a special emphasis on the development of the ideas behind the participants' mathematical artworks and present the results.

Course Concept and Framework

Roughly speaking, our course was divided into three parts. About the first half of the course was devoted to a brief introduction to graph theory, starting with the *seven bridges of Königsberg problem*. Each participant gave a short talk on a basic topic such as special classes of graphs—e.g., complete or bipartite graphs, trees, or networks—and illustrated the introduced notions with their own examples. After each talk, the participants got the opportunity to ask questions and to deepen their newly gained knowledge by solving exercises.

This first mathematical part was interwoven with a second part, consisting of the presentation of examples illustrating already-developed connections between mathematics and arts. These were to serve as inspiration for the artworks that were to be made by the participants later. These examples were chosen to show a broad variety of possibilities, ranging from graphs of polyhedra [8], via trees in the œuvre of Piet Mondrian [6], to

Figure 1: *Left: A participant explains their poster to the group. Right: A poster by a participant, showing two larger tree visualizations, below which is written: "Illustration for instance with string art or acrylic paint, maybe as forest?" The sticky notes suggest, e.g., for styrofoam balls for the vertices, an organic material, or the use of multiple colors. See Figures 4(a) and 4(b) for the final artwork.*

visualizations of a graph's distance matrix [1]. Finally, the third part was dedicated to the development and realization of an individual piece of art incorporating graph-theoretical concepts.

The time dedicated to the introduction of graph theory was twelve lessons of 90 minutes each, while the presentation of examples of the connections between mathematics and arts was three lessons of 90 minutes each. Afterwards, the participants could spend up to nine 90-minute lessons to work on their individual artistic projects. The artworks were shown in an internal exhibition on the last day of the course.

Since schematic representations of traffic networks or flow charts were familiar to our participants, they already had an intuition for basic notions from graph theory such as vertices or edges. Also, they knew of several real-world applications of graphs. We also chose graph theory as the subject is approachable without much previous knowledge. Another factor was that graph theory does not belong to the German school curriculum. Thus, the content was equally new to all participants. Overall, it was well suited for those interested in mathematics as well as for those who came to the course because of the arts.

We realized the course concept during a summer school for gifted German high school students. This kind of summer school is held yearly by the company *Bildung und Begabung gGmbH* [9]. Here, the students can investigate unknown territory without being graded. It consists of up to six courses, each centered in a different discipline, such as mathematics, physics, history, economics, or theater. Every course is accompanied by two supervisors coming from the respective discipline. They are chosen because of their expertise as well as their course concept. The summer schools take place during the German school summer holidays and run for 17 days each. For us, the course consisted of 15 participants aged from 16 to 18, ten women and five men. All of them took math courses, which are mandatory throughout the entire German school curriculum, and had at least a basic arts education of three years, which is equally mandatory.

Development of Ideas and their Realizations

The third part of the course, i.e., work on individual artistic projects, started with a creative session for idea generation. Before the participants got the opportunity to collect ideas, make first sketches, and receive feedback from the group, they were given an overview of the available materials. In creating artwork inspired by graph theory, participants could choose to use wood, nails, threads (for string art), toothpicks and Styrofoam balls, along with more classical materials (e.g., paint and brushes) and technical devices (e.g., set squares or pairs of compasses). Sometimes, the provided collection of materials stimulated creativity. The participants brought laptops, so they could also create a digital piece of art instead of a physical one. Those who were interested were given an introduction to *Processing* [11], cf. a course with *Processing* [5].

In addition to considering artwork created by other artists, the participants used the so-called *gallery method* to collect thoughts, develop ideas, and make first drafts. Following this method, each participant was provided a blank poster and several pens. Participants were given 40 minutes to collect their thoughts and ideas on the poster in any form they liked. Then, they were invited to present their ideas to the group. After doing so, while moving through the gallery, the participants and their supervisors were allowed to place comments written on sticky notes directly on the posters. These comments could include questions, praise, or further ideas. Later, each participant collected their poster and decided how to handle the feedback by keeping those sticky notes whose comments they wanted to incorporate into their artwork and by discarding the rest. See Figure 1 for an image of the setup and an example poster.

After having developed a concept for their piece of art, each participant decided whether they wanted to work physically or digitally and which material to choose for their artwork. Some participants asked for additional material in the form of fabric for embroidery or plywood and a saw to realize their projects, which could be accommodated by *Bildung und Begabung*. During the sessions dedicated to working on the projects, the participants distributed themselves in several rooms available and worked individually. After completing their artwork, the participants wrote artist statements which were displayed in the aforementioned exhibition.

Gallery of Created Artworks

By the end of the course, each participant had created a piece of art on their own. The works and the corresponding complete artist statements are available online [10]. Four members of the group decided to go for a digital project while the others used string art, plywood, colors, embroidery, or fabric. The range of created artworks extended from realizations similar to examples seen in the course, all the way to more individual approaches. Taking the accompanying artist statements into account, some works incorporated a very personal facet of the respective participant. In the following, we present the results of the course and highlight specific learning outcomes achieved by the participants while creating their artworks. Quotes, titles, and explanations stem from the artist statements and have been translated by the authors of this paper.

On the gallery poster shown in Figure 2, participant wrote "*paper boat with nails + string on a board / Eulerian graph / all blue / one boat in the center red.*" From the sketches in the bottom right of the poster, we can see that the participant experimented with different regular tilings. While the corresponding graphs are not Eulerian, the one showing ship-like tiles is. A suggestion on a sticky note read: "*All from one string: It's*

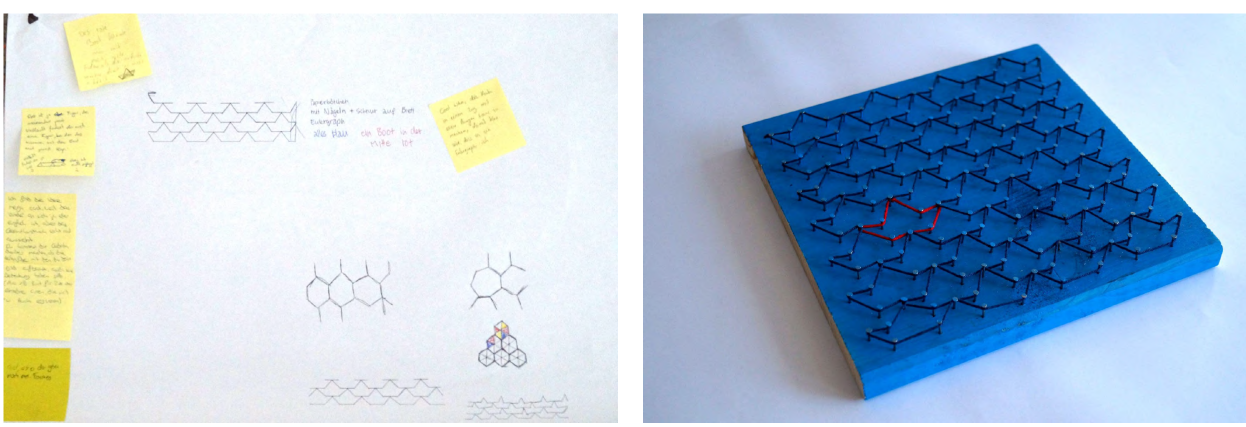


Figure 2: *Gallery poster (left) and artwork (right) by Luzi Soetje: "The Blue Boat." String art, 20 cm × 20 cm, 2022.*

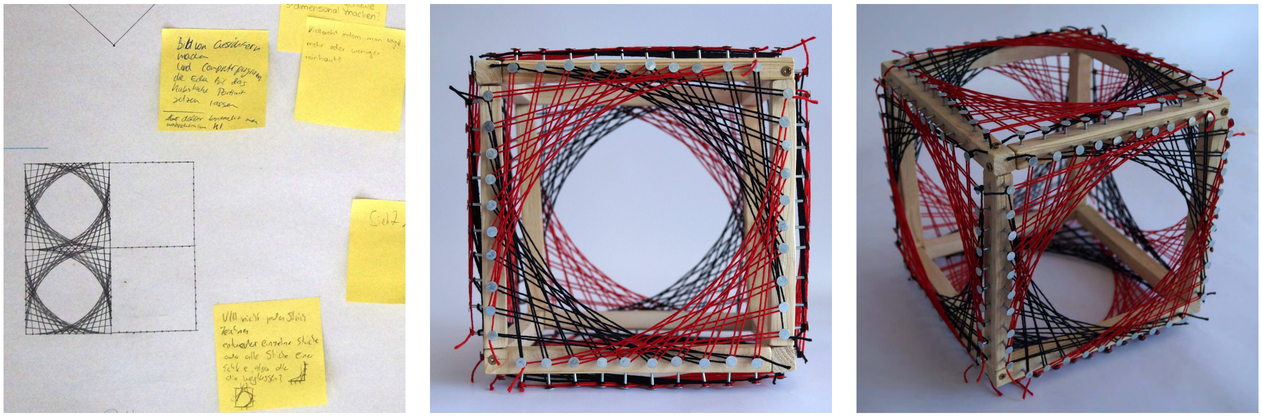


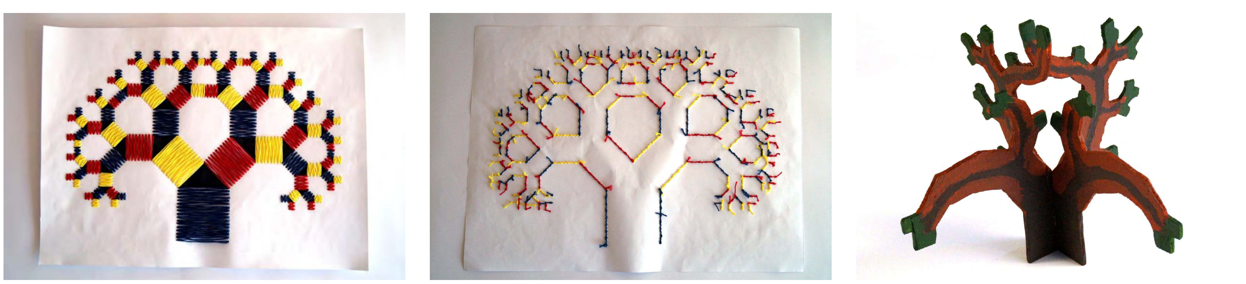
Figure 3: *Section from gallery poster (left) and artwork (center and right) by Ira Hesse: "Trapped in Anxiety and Loneliness." String art, 21 cm × 21 cm × 21 cm, 2022.*

an Eulerian graph." This suggestion was realized and the resulting artwork was created with one blue string representing an Eulerian path along the edges of the graph, except for the small cycle highlighted in red.

A similar observation can be made about the poster and artwork presented in Figure 3. The gallery poster clearly contained a sketch of the work. One of the sticky notes next to it read: "*Maybe do not draw all lines / either omit parts of or all lines of a corner.*" As can be seen, the participant decided to extrapolate the pattern to a cube in 3D. Here, the suggestion was realized via coloring the strings associated with the corners differently. With this came the insight that the corners of the cube are 2-colorable, i.e., there is a coloring with two colors such that no neighboring corners have the same color. Thus, by working on the artistic piece, the participant connected the concepts of a graph coloring with that of the graph of a polyhedron.

Additionally, with regard to their artwork, the creator of "Trapped in Anxiety and Loneliness" wrote: "*With respect to graph theory, [. . .] each thread forms a connected, bipartite graph. Altogether, a net arises spanned around the cube which monopolizes and constricts it. The colors red and black [. . .] let the object appear gloomy which supports the effect of being trapped and creates an uncomfortable tension.*"

To point out a third learning outcome, consider the two different realizations of the *tree of Pythagoras*, as shown in Figure 4. The underlying fractal was built by taking a square, putting a right-angled triangle on one side and drawing squares on the remaining sides of the triangle, iterating the process with these squares. One participant realized the tree by stitching cotton thread on paper (Figures 4(a), 4(b)). In the figure, the resulting tension of the paper is visible, while the front and the back reveal two different views on the fractal.



(a) (b) (c)

Figure 4: *(a) and (b) Artwork based on the gallery poster from Figure 1 by Emilia Nebel: "Primary Color Pythagorean Tree." Thread on colored paper, 37 cm × 50 cm, 2022. (c) Vesselin Matthiesen: "Tree made out of wood." Colored plywood, 20 cm × 20 cm × 17 cm, 2022.*

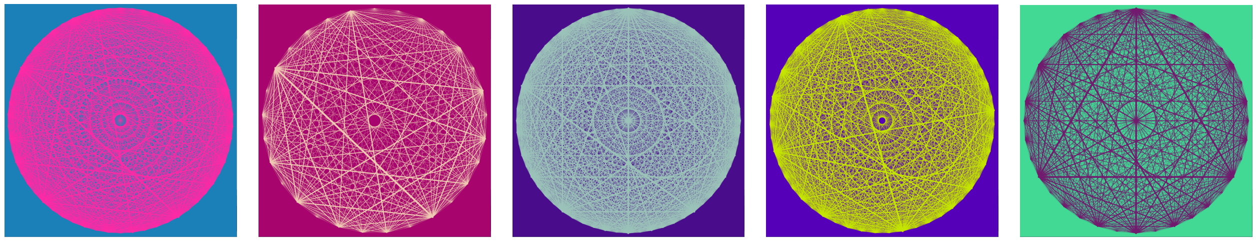


Figure 5: *Jana Reich: "Strobo Mandalas." Digital artwork, 2022.*

In this work, the triangle was chosen to be isosceles. The resulting symmetry of the tree was used as a stopping criterion for the building process, such that the tree is touching, but never overlapping. The other realization of the tree shown in Figure 4(c) was obtained in a different way. Here, the tree was also created from an isosceles triangle, but branches were randomly discarded. The resulting tree was then transferred to plywood and cut out twice. When the participant tried to stick them together at a 90° angle in order to obtain a 3D sculpture, it was not possible. This is because the chosen tree grew back onto the central axis, and as both plywood parts were exactly the same, the two copies collided there when putting them together. The problem was resolved by using symmetry breaking and filing parts of the tree to fit the collision point.

The creator of the five outputs shown in Figure 5 made a piece of *generative art*. It is a playful work that created mandala-like shapes from complete graphs on a randomly chosen number of vertices. In the original implementation, a new shape was shown every second, which created a stroboscopic effect, hence the name "Strobo Mandalas." The appearance of each shape was further emphasized by an audio output, further strengthening the hypnotic quality of the work.

The creator of "Don't PRESS the button" (Figure 6) stated that their work "*is an interactive artwork playing with the desire of a human being to do something forbidden, to make them press the button. The more often the button is pressed, the clearer the structure of this piece gets.*" Similar to the work shown in Figure 5, the outputs were based on a complete graph. Here, the graph had twelve vertices whose inner edges were deleted with a given probability after pressing the button. Furthermore, the emerging polygonal regions were colored randomly following a prescribed color scheme.

One of the graph theory elements introduced in the course was *distance matrices* with a corresponding visualization [1]. The two participants whose works are portrayed in Figure 7 both chose to work with these. The triptych shown in Figures 7(a)–7(c) went for a literal realization. It shows the same distance matrix of a tertiary tree, with three different permutations of the orders of the vertices. The creator of the work shown in Figure 7(d), decided to "*create a tree from colored distance matrices of several different graphs. For the crown of the tree, I mainly used the graphs of trees and binary trees. Thus, my work is largely made up of other trees, which could be said to make it a tree made up of a forest.*"

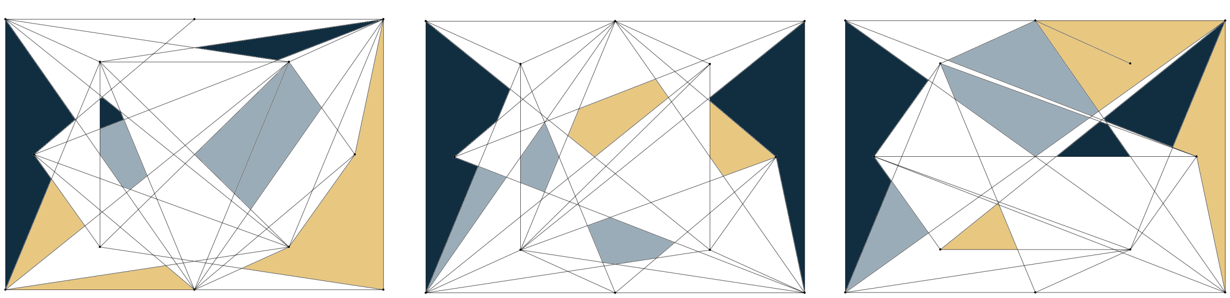
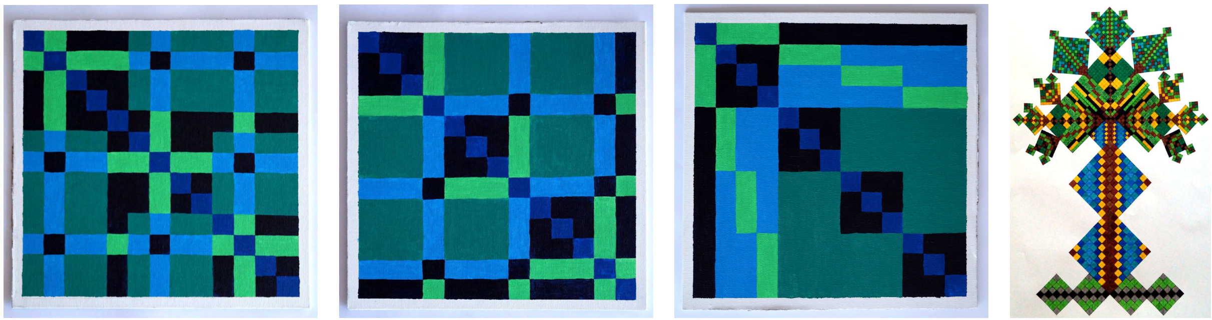


Figure 6: *Sophie Pahmeyer: "Don't PRESS the button." Digital artwork, 2022.*



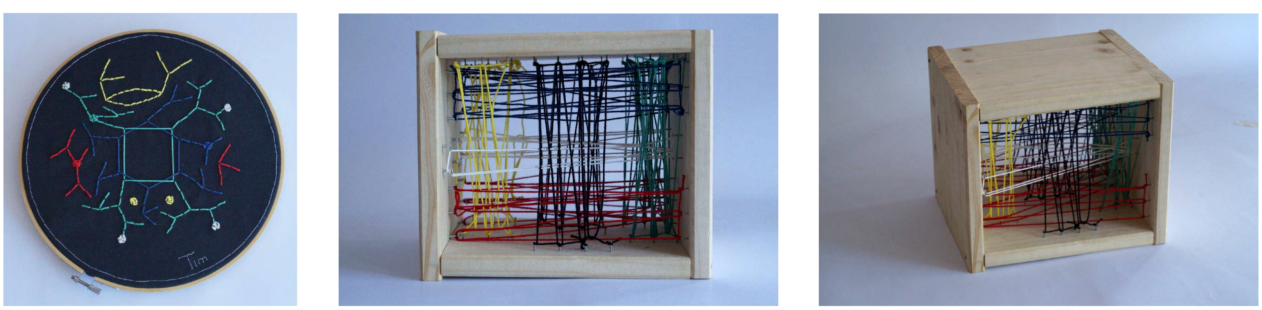
(a) (b) (c) (d)

Figure 7: *(a)–(c) Helen Anderl: "Distance matrices of tertiary trees." Acrylic on canvas, 24 cm × 24 cm each, 2022. (d) Josephine Wehner: "When the tree sprang from the forest." Felt tips on paper, 61 cm × 43.5 cm, 2022.*

The creator of "Stella botanica residua" (Figure 8(a)) realized a concept that combines graph and number theory [2]. Here, a directed graph arises by choosing some $n \in \mathbb{N}$, considering a vertex for each integer $0, \ldots, n-1$ and drawing a directed edge (x, y) if and only if $y \equiv x^2 \mod n$. For $n = 37$ and $n = 41$, the respective graphs have three or four connected components. The participant wrote that "*the cycle of the largest components, a hexagon and a square, are to form the center of the work. The remaining vertices and edges of these components are to entwine the other components, which are placed at the boundary of the work.*" While the embroidery is reminiscent of a constellation (*stella*), the entwining is rather plant-like (*botanica*) and realizes a number theoretic concept (*residua*).

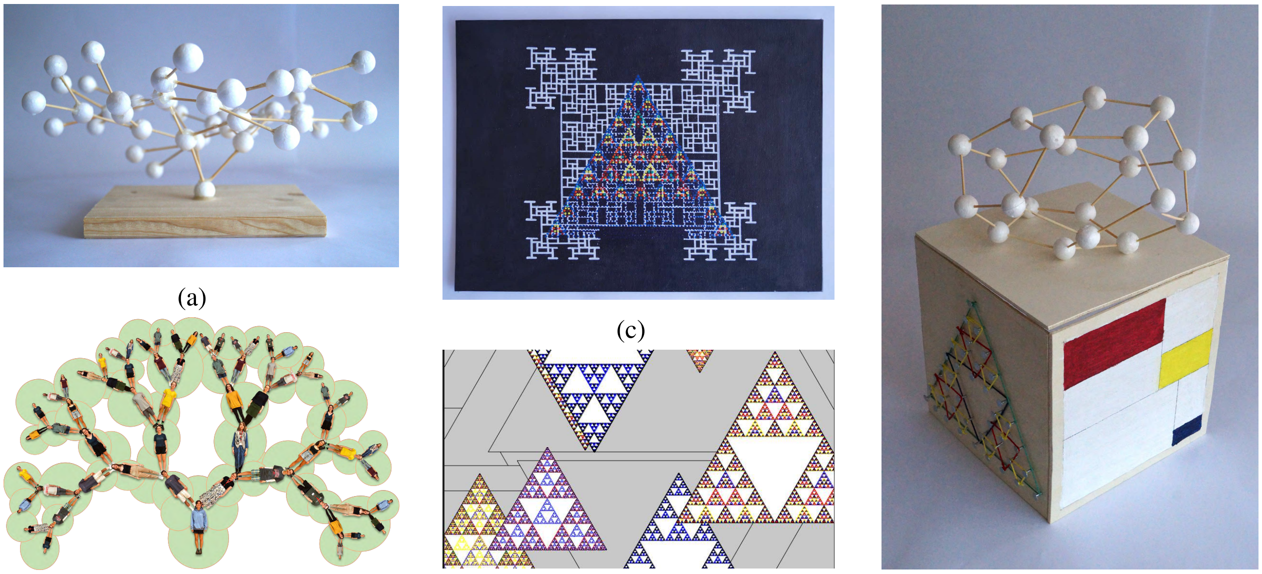
In "Uncertainty" (Figures 8(b) and 8(c)), the theme is bipartite graphs since "*they have—if represented in a suitable way—a simple and ordered structure.*" For an appealing look, the participant decided "*to construct six equally built graphs [. . .] which cross or run against each other.*" Each graph resides in a planar cross-section of the box. The cube-like construction framing the graphs reduced the freedom of movement, hence they had to deviate from bipartite graphs and connect the vertices of one class.

The two artworks shown in Figures 9(a) and 9(b) interpret binary trees in different ways. The piece shown in Figure 9(a) consists of a three-dimensional representation of three binary trees. They are arranged such that they form a vortex which—according to the participant—has similarities to "*a tornado, because it is bigger and more irregular and fits more to the rough structure of my project. In contrast, a whirlpool is very regular and too uniform to resemble my 3BiT–Tree spiral.*"



(a) (b) (c)

Figure 8: *(a) Tim Burdack: "Stella botanica residua." Cotton thread on fabric, 25 cm in diameter, 2022. (b) and (c) Mareike Pfeilsticker: "Uncertainty." String Art, 22 cm × 20 cm × 20 cm, 2022.*



(a) (b) (c) (d) (e)

Figure 9: *(a) Nolan Muniz Fernandez: "3BiT–Tree spiral." Styrofoam, wood, and string, 30 cm × 30 cm × 15 cm, 2022. (b) Isabella Kaindl: "A tree says more than a thousand numbers?!" Collage, 2022. (c) Maja Sixtbauer: "Fractals over Fractals." Acrylic on canvas, 24 cm × 30 cm, 2022. (d) Paul Kästner: "Three Kinds of Vertices." Digital art, 2022. (e) Leo Sprengard: "Cubus Artis". String art on colored plywood, 20 cm × 20 cm × 20 cm, 2022.*

In the artwork shown in Figure 9(b), the edges are not drawn as straight line segments, but consist of photographs taken from the participants and supervisors. Here, the individuals form an ordered and symmetric graph. The represented tree is emphasized by green disks relating the mathematical tree to a biological one in order to make the artwork accessible for observers without a mathematical background. The creator wrote: "*Through illustration, I try to give an insight into graph theory through art accessible to everyone. Although a lot of important information can be lost by a visualization, I have tried via this visualization to create a connecting and interdisciplinary work through art.*"

The artworks depicted in Figures 9(c) to 9(e) show three different interpretations of the *Sierpinski triangle*, another object that was discussed in the course. While the two pieces in the center combine the Sierpinski triangle either with a tooth pick sequence (Figure 9(c)) or with several other Sierpinski triangles (Figure 9(d)), the artwork shown in Figure 9(e) incorporates it into a sculpture that captures two other mathematical objects from the course: the graph of a dodecahedron [8] and a k-d tree [6]. In contrast to the two Sierpinski triangles in Figures 9(c) and 9(d), the one in Figure 9(e) does not consider a coloring of the vertices, but of the edges.

Summary and Conclusions

In 2019, the course concept above was carried out in a similar context, but focusing on the artwork of Paul Klee [7]. Since Klee's interpretation of color gradients is very close to the mathematicians' point of view—as Klee explains in [4]—it was natural to relate his ideas to multivariate calculus. Like graph theory, multivariate calculus is not part of the school curriculum. However, it turned out to be more difficult for the students than expected even though several concepts were familiar from school. In contrast to graph theory, visualizations of various concepts from multivariate calculus are difficult to be achieved by hand. Even though Klee himself connected his art to mathematics, this level of abstraction was comparably high, and the participants had trouble relating to it. With the choice of graph theory, we found a subject that is overall more relatable, possibly also to other student groups, younger or older than high school students. We recommend future supervisors of this course concept to carefully choose a mathematical topic easy to access and sufficiently far away from the school curriculum. Note that we, the two course leads, are both mathematicians by training. Thus, while being able to support the mathematical learning of the students well, we were only able to support the artistic learning and skills development on our own basis of an autodidact's knowledge.

With the given retrospective of both the concept of our summer school course and its results, we hope to highlight the opportunities of a course on mathematics and arts. We hope that our reflections on the course execution help the reader to establish a similar offering in their own setting. In particular the shift to the graph theoretical content creates a course that is not only more accessible, but also somewhat age-independent. While our participants were 15–18 years old, the rich examples graph theory offers make the subject well suitable for other age groups, too. We see tremendous potential in adapting such a course to a mathematics club at school, a project day or week, and university courses.

In our work with participants that were reluctant to engage with either mathematics or art, we have observed an increase in the willingness to tackle the content, particularly through the opportunity to engage in self-selected and self-designed projects. This article may help to give more people the opportunity to discover mathematics in a creative and individual way. While we feel that the course has improved compared to the first iteration, the outcomes have not been formally evaluated. This is left for a subsequent version.

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