Supplement 1

Suppose an object has a group of symmetries G of size k, and there are m locations on the object which are permuted by the symmetries. The cycle index of such an object is [10, Sec. 6.8]

$$Z_G(x_1, x_2, \ldots, x_m) = \frac{1}{k} \sum_{g \text{ in } G} x_1^{c_1(g)} x_2^{c_2(g)} \ldots x_m^{c_m(g)},$$

where $c_i(g)$ is the number of cycles of the symmetry g which have size i.

For an object with m locations and only m-fold rotational symmetry, the cycle index is [5]

$$C_m(x_1, \ldots, x_m) = \frac{1}{m} \sum_{d \text{ divides } m} \phi(d) x_d^{m/d},$$

where $\phi(d)$ is the Euler totient function which counts the number of integers between 1 and d relatively prime to d. For instance, if $m = 16$,

$$C_{16}(x_1, \ldots, x_{16}) = \frac{1}{16}\left(x_1^{16} + x_2^8 + 2x_4^4 + 4x_8^2 + 8x_{16}\right),$$

corresponding to:

- the identity, which has 16 cycles each of size 1 (the outer ring in Figure 4),
- rotation by 180°, which has 8 cycles each of size 2 (the second-largest ring in Figure 4),
- rotations by 90° and 270°, which each have 4 cycles each of size 4 (ring 3 in Figure 4),
- rotations by 45°, 135°, 225°, and 315°, which each have 2 cycles each of size 8 (ring 4 in Figure 4), and
- the 8 rotations of multiples of 22.5° not yet listed, which each have a single cycle of size 16 (the inner ring in Figure 4).

For an object with m locations and m-fold dihedral symmetry, where m is an even number, the cycle index is [5]

$$D_m(x_1, \ldots, x_m) = \frac{1}{2m}\left(\sum_{d \text{ divides } m} \phi(d) x_d^{m/d} + \frac{m}{2} x_2^{m/2} + \frac{m}{2} x_1^2 x_2^{m/2-1}\right).$$

For example, if $m = 16$, as in the 16-strand Kongō Gumi braid,

$$D_{16} = \frac{1}{32}\left(x_1^{16} + x_2^8 + 2x_4^4 + 4x_8^2 + 8x_{16} + 8x_2^8 + 8x_1^2 x_2^7\right).$$

This corresponds to the 16 rotations listed above, plus:

- 8 reflections passing between the colored locations, each of which has 8 cycles each of size 2 (the 8 rings of Figure 5), and
- 8 reflections passing through two colored locations, each of which has 7 cycles of size 2 and 2 cycles of size 1 (the 8 rings of Figure 6).

We also need the cycle index for the ways to permute q colors, which is [5]

$$S_q(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{d_1 + 2d_2 + \cdots + q d_q = q} \frac{q!}{d_1! \ldots d_q!} x_1^{d_1} \left(\frac{x_2}{2}\right)^{d_2} \ldots \left(\frac{x_q}{q}\right)^{d_q}.$$

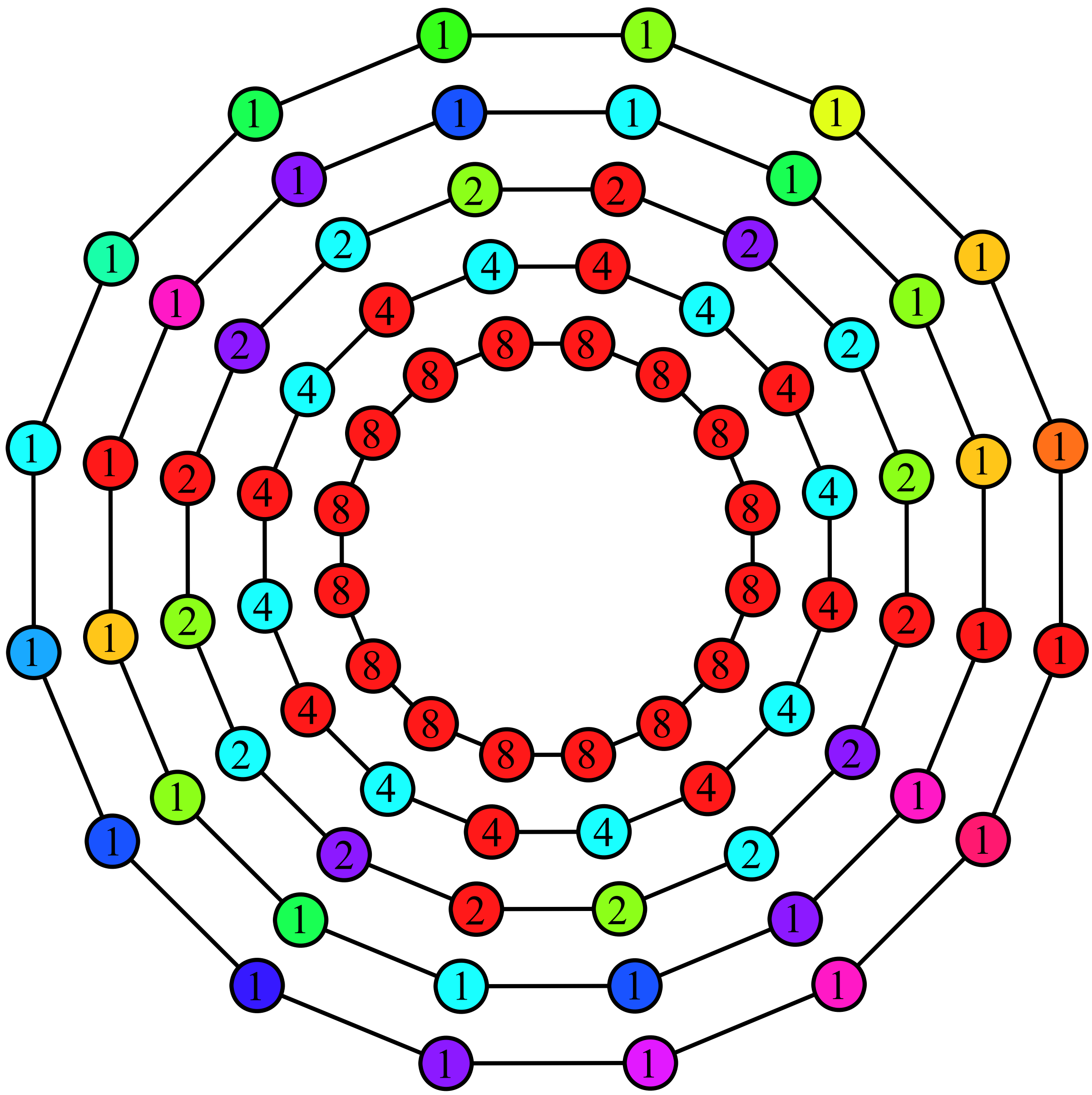


Figure 4: *Cycles corresponding to rotations of a 16-strand braid around its axis. Each ring represents a set of symmetries that share a set of cycles. Each color within the ring represents a cycle for that set of symmetries. (The colors of each ring are not related to the colors of other rings.) The numbers indicate the number of symmetries with that cycle.*

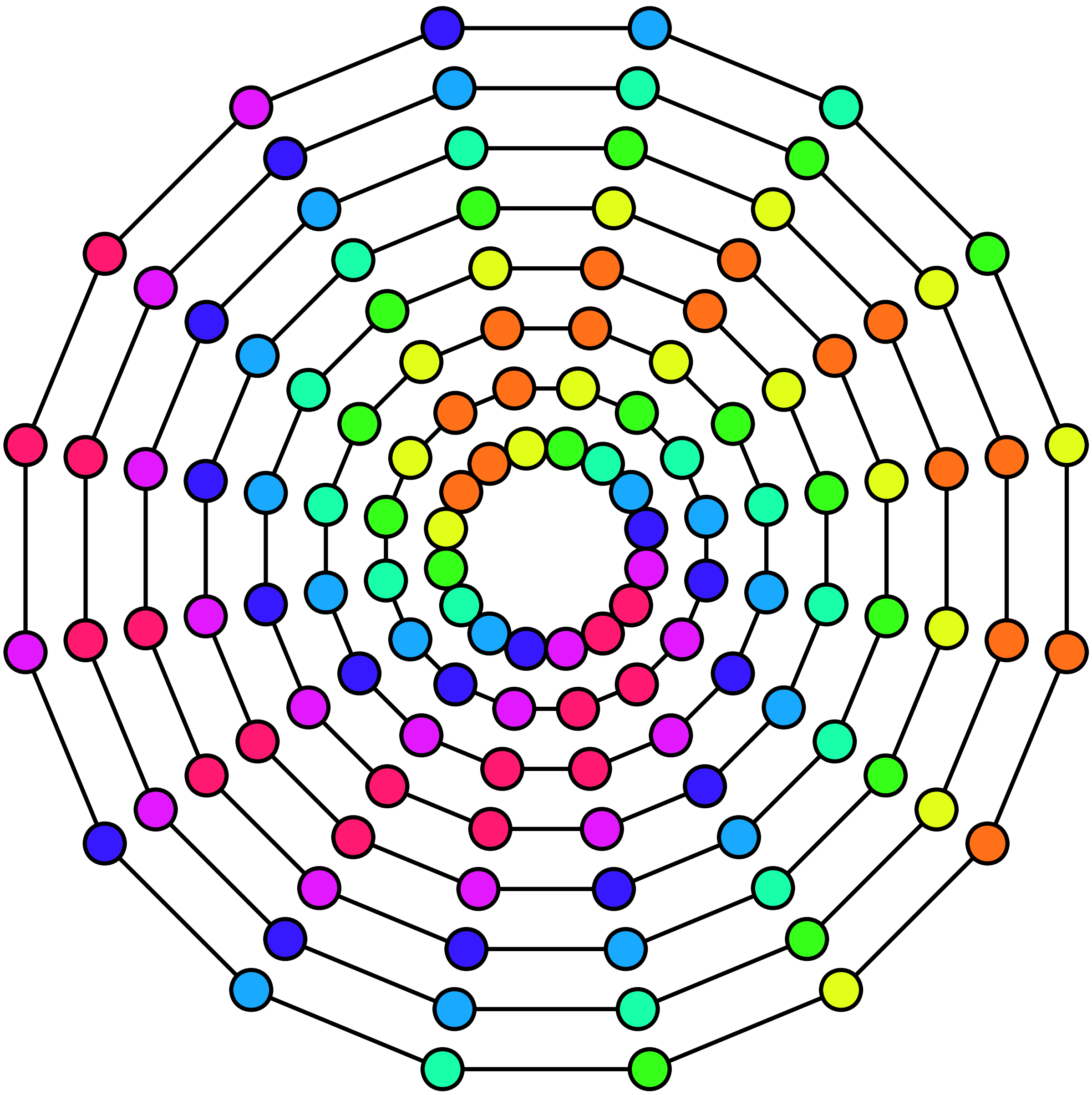


Figure 5: *Cycles corresponding to reflections of the braid passing between colored locations. Each ring represents a single symmetry and its set of cycles; each color represents a cycle.*

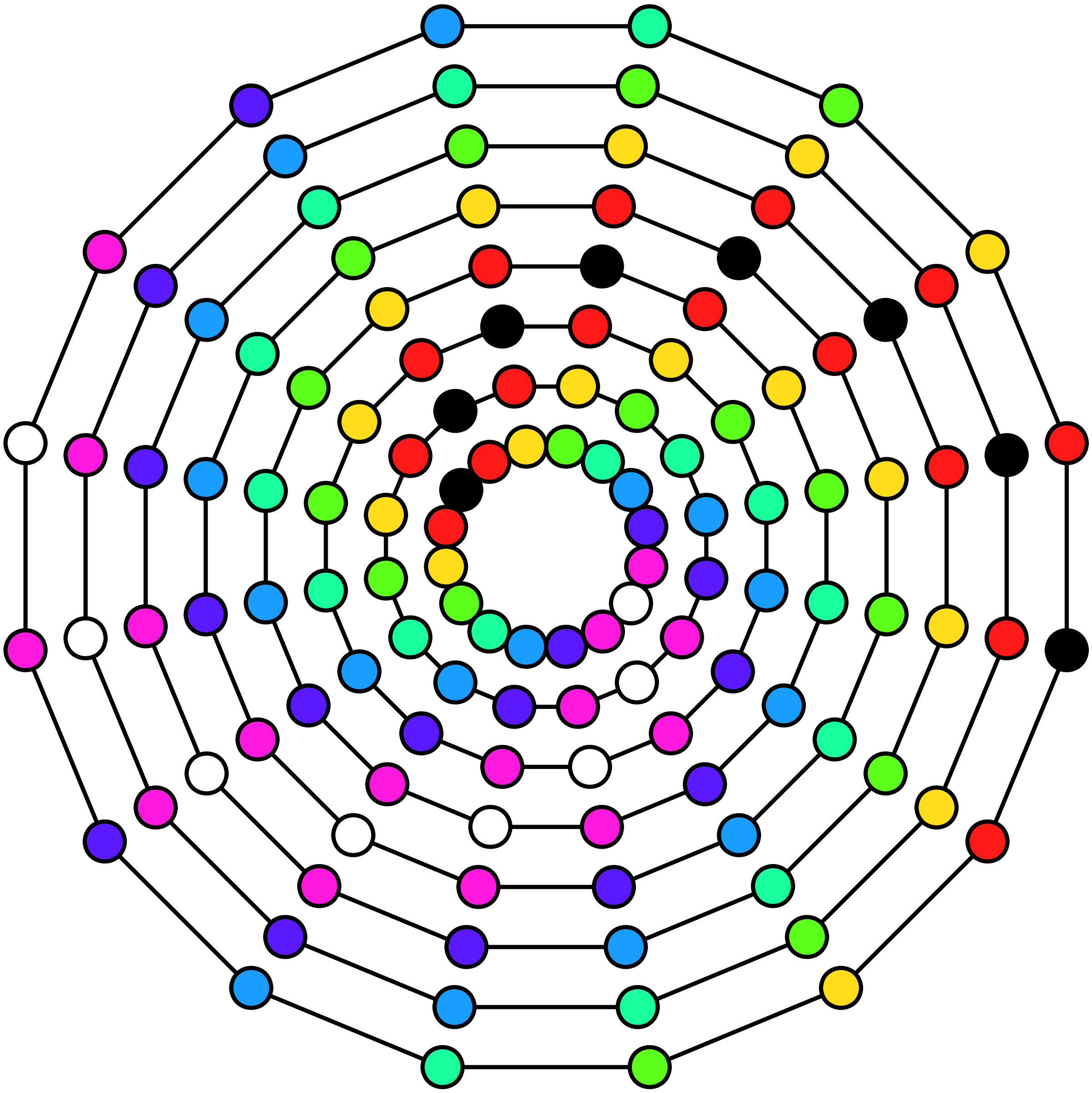


Figure 6: *Cycles corresponding to reflections of the braid passing through two colored locations (the black and white dots). Each ring represents a single symmetry and its set of cycles; each color represents a cycle.*

For example, with $q = 2$ colors we have

$$S_2(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2\right)$$

and with $q = 4$ colors we have

$$S_4(x_1, x_2, x_3, x_4) = \frac{1}{24}\left(x_1^4 + 6x_1^2 x_2 + 8x_1 x_3 + 3x_2^2 + 6x_4\right).$$

A simplified form of de Bruijn's Theorem now says:

Theorem (de Bruijn [5])**.** *If an object has m locations which can each be colored with one of q different colors, and a group G of symmetries with cycle index $H_m(x_1, \ldots, x_m)$, then the number of non-equivalent ways to color the object is*

$$H_m\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}\right) S_q\left(e^{1(z_1+z_2+\ldots)}, e^{2(z_2+z_4+\ldots)}, \ldots, e^{q(z_q+z_{2q}+\ldots)}\right)$$

evaluated at $z_1 = z_2 = \cdots = z_m = 0$.

For example,

$$D_{16}\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{16}}\right) S_2\left(e^{1(z_1+z_2+\ldots)}, e^{2(z_2+z_4+\ldots)}\right)$$

$$= \frac{1}{32}\frac{\partial^{16}}{\partial {z_1}^{16}} S_2 + \frac{1}{4}\frac{\partial^9}{\partial {z_2}^7 \partial {z_1}^2} S_2 + \frac{9}{32}\frac{\partial^8}{\partial {z_2}^8} S_2 + \frac{1}{16}\frac{\partial^4}{\partial {z_4}^4} S_2 + \frac{1}{8}\frac{\partial^2}{\partial {z_8}^2} S_2 + \frac{1}{4}\frac{\partial}{\partial z_{16}} S_2$$

$$= 1125\left(e^{z_1+z_2+z_3+z_4+z_5+z_6+z_7+z_8+z_9+z_{10}+z_{11}+z_{12}+z_{13}+z_{14}+z_{15}+z_{16}}\right)^2$$
$$+ 37e^{2z_2+2z_4+2z_6+2z_8+2z_{10}+2z_{12}+2z_{14}+2z_{16}+2z_{18}+2z_{20}+2z_{22}+2z_{24}+2z_{26}+2z_{28}+2z_{30}+2z_{32}}$$

which evaluated at $z_1 = z_2 = \cdots = z_{16} = 0$ gives 1162.

The more complete version of de Bruijn's Theorem says:

Theorem (de Bruijn [4, Special case of Thm. 1])**.** *If an object has m locations which can each be colored with one of q different colors, and a group G of symmetries with cycle index $H_m(x_1, \ldots, x_m)$, then the number of non-equivalent ways to color the object with the first color in m_1 locations, the second color in m_2 locations, and so on, is the coefficient of $w_1^{m_1} w_2^{m_2} \ldots$ in*

$$H_m\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}\right) S_q\left(\eta_1, \ldots \eta_q\right)$$

where η_k is the power series expansion in x of

$$e^{k(z_k x + z_{2k} x^2 + \ldots)}$$

with x^j replaced by w_j^t for each j, evaluated at $z_1 = z_2 = \cdots = z_m = 0$.

For example,

$$D_{16}\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{16}}\right) S_4\left(\eta_1, \eta_2, \eta_3, \eta_4)\right)$$

$$= \frac{1}{32}\frac{\partial^{16}}{\partial {z_1}^{16}} S_4 + \frac{1}{4}\frac{\partial^9}{\partial {z_2}^7 \partial {z_1}^2} S_4 + \frac{9}{32}\frac{\partial^8}{\partial {z_2}^8} S_4 + \frac{1}{16}\frac{\partial^4}{\partial {z_4}^4} S_2 + \frac{1}{8}\frac{\partial^2}{\partial {z_8}^2} S_4 + \frac{1}{4}\frac{\partial}{\partial z_{16}} S_4$$

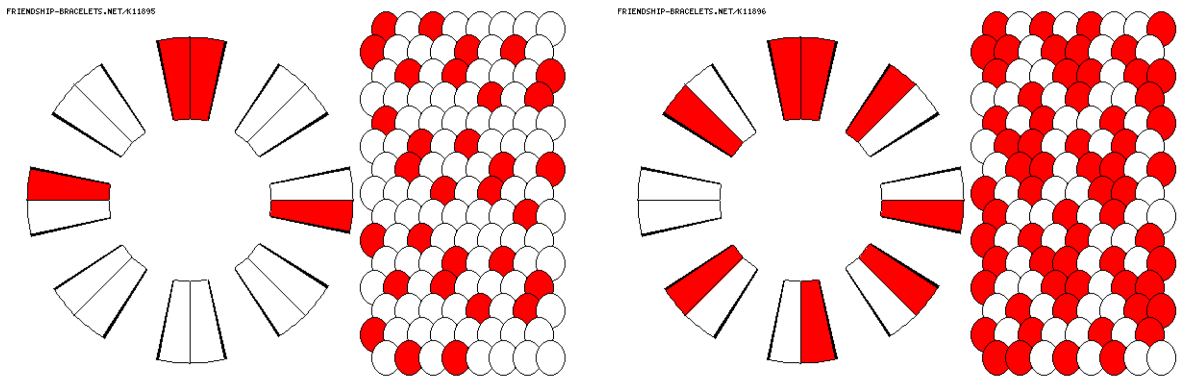
evaluated at $z_1 = z_2 = \cdots = z_{16} = 0$ gives

$$= 83488{w_4}^4 + 788865 w_3 {w_4}^2 w_5 + 316701 {w_3}^2 {w_5}^2 + 474138 w_2 w_4 w_5^2 + \cdots$$

from which coefficients come the values in Supplement 3.

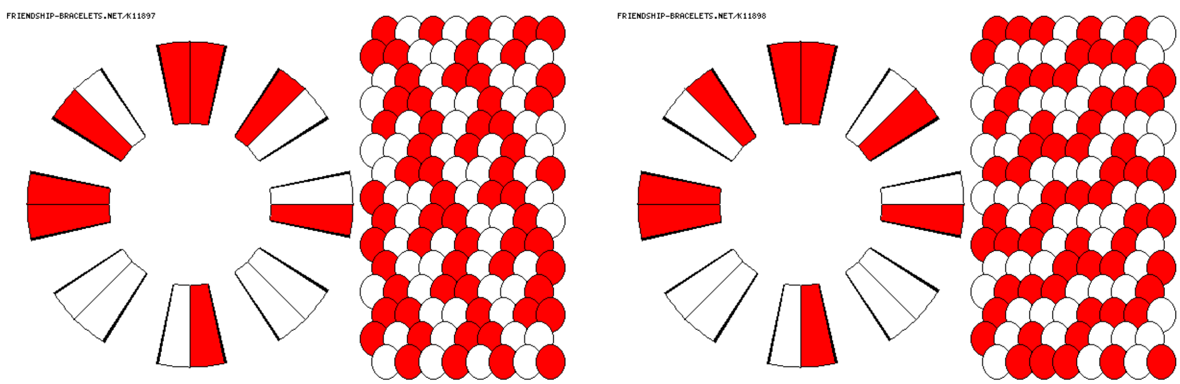
Supplement 2

The follow diagrams show the braids missing from Rosalie Neilson's book [8]. (All pictures generated by the author using Friendship-Bracelets.net.)



(a) **(b)**

Figure 7: *(a) The missing 4-spot braid: 1.2//6.14/ in Neilson's notation.*
(b) The first missing 8-spot braid: 1.2.9/3.12/6/7.15 in Neilson's notation.



(a) **(b)**

Figure 8: *(a) The second missing 8-spot braid: 1.2.9/3/6.13.14/15 in Neilson's notation.*
(b) The third missing 8-spot braid: 1.2.9/4/6.13.14/16 in Neilson's notation.

Supplement 3

The following table gives the number of non-equivalent 16-strand Kongō Gumi patterns with exactly four different colors, classified by the distribution of strands.

Number of strands of each color				Number of patterns
Color A	B	C	D	
4	4	4	4	83488
3	4	4	5	788865
3	3	5	5	316701
2	4	5	5	474138
1	5	5	5	63231
3	3	4	6	526610
2	4	4	6	395500
2	3	5	6	630840
1	4	5	6	315420
2	2	6	6	80053
1	3	6	6	105252
3	3	3	7	100380
2	3	4	7	450660
1	4	4	7	112805
2	2	5	7	135450
1	3	5	7	180180
1	2	6	7	90160
1	1	7	7	6630
2	3	3	8	113155
2	2	4	8	85077
1	3	4	8	112665
1	2	5	8	67620
1	1	6	8	11410
2	2	3	9	37695
1	3	3	9	25095
1	2	4	9	37590
1	1	5	9	7560
2	2	2	10	3941
1	2	3	10	15036
1	1	4	10	3843
1	2	2	11	2079
1	1	3	11	1386
1	1	2	12	371
1	1	1	13	21

The next table gives the number of non-equivalent 16-strand Kongō Gumi patterns with exactly three different colors, classified by the distribution of strands.

Number of strands of each color			Number of patterns
Color A	B	C	
5	5	6	31920
4	6	6	26734
4	5	7	45150
3	6	7	30100
2	7	7	6630
4	4	8	14417
3	5	8	22575
2	6	8	11410
1	7	8	3235
3	4	9	12565
2	5	9	7560
1	6	9	2520
3	3	10	2611
2	4	10	3843
1	5	10	1512
2	3	11	1386
1	4	11	693
2	2	12	212
1	3	12	231
1	2	13	56
1	1	14	8

The values for two colors and one color are given in Table 1 of the paper.