Adjustable Duotone Mosaic Tile Brightness via Bézier Boundaries

Amy E. Wendt

Electical and Computer Eng. Dept., University of Wisconsin-Madison; aewendt@wisc.edu

Abstract

Duotones are a graphic design mainstay with application to print media and textiles. In the case of mosaic images constructed with duotone tiles arranged in a grid, the "brightness" of each tile is tuned by adjusting the fractional areas within the tile occupied by a dark and a light color, respectively. Use of duotone tile types containing symmetries provides opportunities for algorithmic approaches to systematically representing pixel brightness, facilitating automation as an aid in constructing mosaic representations of images. A parametric representation using Bézier curves is described here for a type of Truchet-like square tiles each with a pair of opposite corners depicted in a contrasting color. A single parameter controls tile brightness by smoothly distorting curved boundaries between the color regions. Illustrations of the method include mosaics depicting simple gradients and more complex images. Additional considerations in some textile applications, including patchwork quilts and knitting, are also explored.

Introduction

Duotone images are made up of just two colors, either black and white or another pair of high contrast hues, with gradients produced by varying the local proportion of the two colors. A commonly used example of creating gradients in duotone print images is the use of halftone dots, where the proportion of two colors is varied through the use of a regular grid of circular dots overlaying a contrasting background, with the dot radius as a control parameter. Bosch and collaborators [2, 1, 12] have elevated duotone art to new levels through the use of sets of two-colored tiles in which specified rules and optimization algorithms are applied to create figurative mosaics.

In every civilization and culture, colored tilings and patterns appear among the earliest of decorations [7]. This paper extends the implementation of high contrast mosaic gradients and images through the introduction of an approach involving a type of duotone [10] square tile motif related to those described by Sébastien Truchet in his seminal paper published in 1704 [19]. Truchet's scholarly treatise enumerating the symmetries and decorative patterns produced through the arrangement of two-color diagonally divided tiles has led to several distinct members of this class of tiles bearing his name.

A popular variant and the focus of this paper is a tile consisting of a square overlaid with a pair of quarter circles centered in diagonally opposite corners and with radius equal to half the length of the sides of the square [18], so that the circular arcs on adjacent tiles connect to form smooth contours. The circular arcs may be outlines (*contour*-style) or filled to form contrasting quarter circle wedges (*duotone*-style). This motif is known as Smith tiles in recognition of the 1987 paper in which C. S. Smith and P. Boucher [18] establish the association to the tile symmetries and patterns described by Truchet. However, the Truchet-style circular arc motif is also widely recognized as a popular pieced quilt design known as “Snake Trail” (among other names), with a first documented publication in 1910 [3, 17], predating Smith's. In the duotone form, two possible color arrangements (forming a *counterchange* [10] tile set) and two rotational orientations are possible, resulting in four distinct tiles, shown in Figure 1. Closed color domains bounded by smooth curves result when a mosaic is created through an arrangement of tiles, oriented with the colors on each tile edge matching those of the abutting edge of the neighboring tile [4].

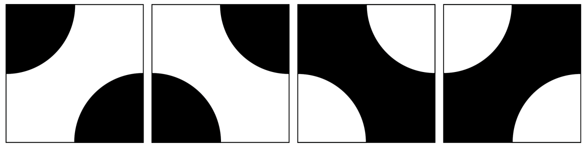

Figure 1: *The set of four traditional-style black and white duotone Smith tiles.*

Programming color boundaries via Bézier Curves

Classic Truchet tiles have black and white triangular regions of equal area, and the tiles with quarter circle corners shown in Figure 1 have ~39.4% and ~60.6% white area, respectively. Neither set is therefore suitable for the construction of figurative mosaics, due to their limited brightness variation. Mathematical artists have developed methods to modify Truchet tiles using Bézier curves to form sophisticated and aesthetic designs [13, 15]. However, it was the introduction by Bosch of a mathematical framework for programming tile brightness through systematic adjustment of the color boundary that provided the basis for creating figurative mosaics constructed of Truchet style tiles [2, 1, 12]. (Multi-scale versions of *contour*-style Truchet tiles [4, 5] have also been exploited to produce image mosaics.) In this paper, I describe the use of Bézier curves as a tool for adjusting *curvature* of color boundaries to achieve a continuous and wide dynamic range in tile brightness.

The cubic Bézier curve provides a parametric representation of the (x, y) coordinates of the curve that separates white and black (or other pair of contrasting colors) regions of the tile, in the form of a cubic polynomial in the parameter t [6, 8]. Four “control points” define the curve, where two of the points are at fixed locations at the ends of the curve, the point where the curve intersects the outer boundary of the square tile at the edge midpoint. The general form of the Bézier curve based on four control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ is:

$$\mathbf{B}(t) = (1-t)^3\mathbf{P}_0 + 3(1-t)^2t\mathbf{P}_1 + 3(1-t)t^2\mathbf{P}_2 + t^3\mathbf{P}_3,$$

where $\mathbf{B}(t) = (x(t), y(t))$ represents the coordinates of the points along the curve, and the range of the parameter t is the interval $[0, 1]$.

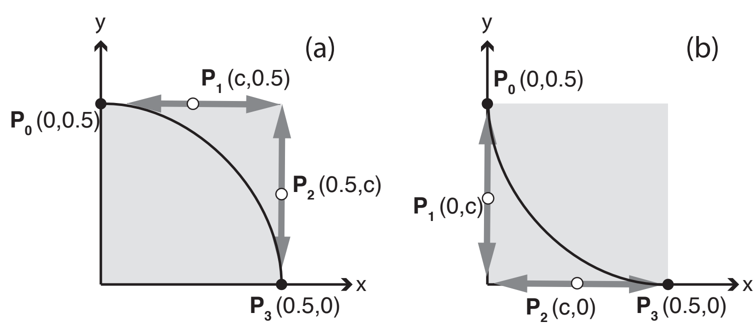

Figure 2: *Cubic Bézier curves to be employed as color boundaries in tile corner quadrants. Bézier control point positions two fixed at (0,0.5) and (0.5,0), and two set by variable $c \in [0, 0.5]$ as shown, producing (a) positive curvature (convex) and (b) negative curvature (concave).*

Treating the fractional white area of a 1×1 square tile as a proxy for tile brightness on a scale of 0 to 1, we calculate the area of the region bounded by the lower edge of the tile and the Bézier curve that separates light and dark regions. In choosing a system for adjusting the curvature of the internal color boundaries on the tile, several constraints apply: 1) The Bézier curve endpoints will remain at the midpoints of the line segments forming the edges of the tiles, a necessary condition to ensure color matching without discontinuity at the boundaries between neighboring tiles. 2) Mirror symmetry will be maintained across both diagonals bisecting the tile. 3) A final constraint is that all of the points on the Bézier curve must remain in the tile quadrant (highlighted in light gray in Figure 2) defined by the nearest tile corner and the two curve endpoints.

To maximize the range of tile brightness, two distinct types of curve are used to form color boundaries, with negative (concave corner) and positive (convex corner) curvature, respectively. The two types employ different sets of Bézier control points as shown in Figure 2(a) and (b), with the fixed endpoints in both cases located at the midpoints of adjacent tile edges. First we consider the category of convex curves, introduced by Kirdeikis [11]. For a 1×1 square tile in the first quadrant of the x-y plane with its lower left corner at the origin, the curved boundary control points are $\mathbf{P}_0 = (0, 0.5)$, $\mathbf{P}_1 = (c, 0.5)$, $\mathbf{P}_2 = (0.5, c)$ and $\mathbf{P}_3 = (0.5, 0)$. It has been shown that using a value of Bézier control parameter $c \simeq 0.276$ produces the best approximation to the quarter circle, with average radial deviation of $\sim$0.02% [9, 14]. The programmable tile system uses this set of control points for tiles with convex corners, with c as a variable in the range 0 to 0.5 (as indicated by the gray arrows in Figure 2) to control the curvature of the color boundary. The control points to produce concave corners are $\mathbf{P}_0 = (0, 0.5)$, $\mathbf{P}_1 = (0, c)$, $\mathbf{P}_2 = (c, 0)$ and $\mathbf{P}_3 = (0.5, 0)$. Figure 3 includes examples of the four types of flexible tiles for representative values of c.

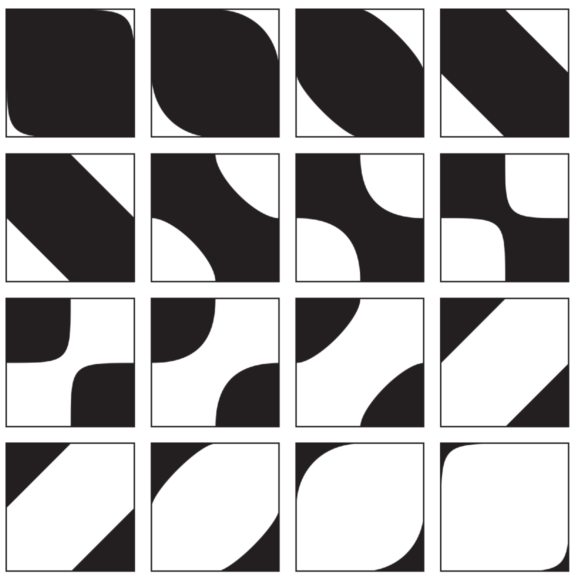

Figure 3: *Each of the four rows shows one of the four file types, and each row contains a progression of tiles of representative brightnesses from lowest to highest for the tile category.*

Mapping Bézier Control Parameters to Tile Brightness

To use the adjustable tile system to create a mosaic of a rasterized image, the brightness level of each image pixel is mapped to the corresponding white fraction of the area of the corresponding mosaic tile. In evaluating tile brightness (on a scale of 0 to 1, with 0 being all black and 1 being all white), we start with the white Bézier corners on black tiles (low brightness) and then appeal to symmetry to extend the result to the counterchanged tiles with black Bézier corners on white (high brightness). For the convex tile type, the minimum brightness value for the full tile (including identical contributions to the white area from two corners of the symmetric tile) is 0.25. This minimum occurs for the case $c = 0$, when the Bézier curve becomes a straight line between the fixed endpoints as shown in Figure 3 (left tile in the second row from the top), a motif found in Kufic calligraphy and Persian ornamental design [16]. Similarly, for the counterchanged (black on white) convex tile, the maximum brightness is 0.75 (rightmost tile in the third row). The introduction of tiles with concave corners in this paper is motivated by the desire to extend the dynamic range of the brightness scale.

For the convex corner, substituting the control points (Figure 2(a)) into the expression for $\mathbf{B}(t)$ yields simplified forms for the coordinates of the curved edge, $\mathbf{B}_{\text{vex}}(t) = (x_{\text{vex}}(t), y_{\text{vex}}(t))$:

$$x_{\text{vex}}(t) = 3c(1-t)^2 t + 1.5(1-t)t^2 + 0.5t^3$$
$$y_{\text{vex}}(t) = 0.5(1-t)^3 + 1.5(1-t)^2 t + 3c(1-t)t^2$$

The area of the white corner defined by the parametric curve becomes $\int_0^{0.5} y\,dx = \int_0^1 y(t)x'(t)dt$. For the tiles with white convex Bézier corners, the tile brightness as a function of the control parameter c, $b_{\text{vex}}(c)$ is defined as twice the area under the curve of a single corner, since the full tile has two white corners. Integration yields the convex tile brightness as a function of the Bézier curve parameter $c \in [0, 0.5]$:

$$b_{\text{vex}}(c) = 0.25 + 0.6c - 0.3c^2$$

The same procedure can be applied to determine the brightness of the tiles with concave white corners, by now using the set of four control points that produce negative curvature, as defined above and in Figure 2(b). The coordinates that define the concave Bézier curve and the corresponding brightness for $c \in [0, 0.5]$ are:

$$x_{\text{cav}}(t) = 3c(1-t)t^2 + 0.5t^3$$
$$y_{\text{cav}}(t) = 0.5(1-t)^3 + 3c(1-t)^2 t$$
$$b_{\text{cav}}(c) = 0.025 + 0.3c + 0.3c^2$$

Extending the definition of brightness to the tiles with black Bézier corners and changing variables to replace

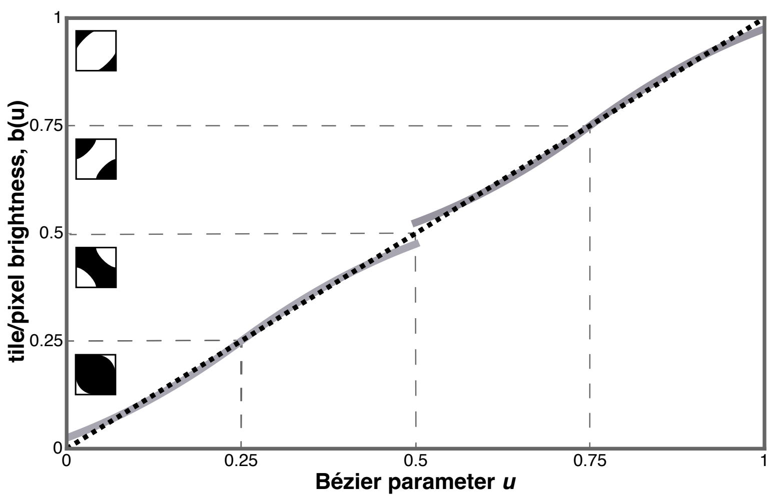

Figure 4: *The gray curve represents an explicit calculation of the fractional white area of the tile b in terms of the Bézier control parameter u, and the dotted line is the linear approximation $b(u) = u$. The boundaries of the ranges of u and b for each of the four tile types are demarcated by dashed lines.*

c with a new Bézier control parameter u that varies from 0 to 1 continuously across the four tile types, a mapping can be constructed between u and tile brightness $b(u)$, represented by the gray curve in Figure 4:

$$b(u) = \begin{cases} 0.025 + 0.6u + 1.2u^2, & 0 \le u \le 0.25 \\ -0.125 + 1.8u - 1.2u^2, & 0.25 \le u \le 0.5 \\ 0.525 - 0.6u + 1.2u^2, & 0.5 \le u \le 0.75 \\ -0.825 + 3u - 1.2u^2, & 0.75 \le u \le 1 \end{cases}$$

The inverse mapping $u(b)$ can then be used to construct mosaic tiles based upon pixel image brightness data. The minimum and maximum attainable tile brightnesses are $b_{\text{min}}(0) = 0.025$ and $b_{\text{max}}(1) = 0.975$.

The brightness $b(u)$ is a piecewise quadratic function of Bézier control parameter u. We *could* therefore solve the quadratic equation directly for each tile type to invert the expression, producing a piecewise analytic function for the Bézier control parameter as a function of brightness, $u(b)$. In making a mosaic of an image, $u(b)$ is used to construct a tile based on brightness b of the pixel in the original image that it will replace. However, as seen in Figure 4, the exact mapping $u(b)$ has gaps for brightness values in the vicinity of $b = 0$, $b = 0.5$ and $b = 1$, corresponding to discontinuities in $b(u)$. Noting that these gaps are relatively small, we address the shortcoming by replacing the exact mapping with the linear approximation $u(b) = b$ (dotted line in Figure 4), which produces an unexpectedly good approximation to the true function. The slight distortion of the mapping by the linear approximation eliminates the gap and returns a value of u (and, therefore tile specification) for every possible pixel brightness. Considering the application to the production of relatively low resolution mosaics, the compromise of using the linear approximation seems quite acceptable, and offers the advantage of simplicity compared to a more accurate mapping.

Example Mosaics

Mosaic examples are presented in Figures 5, 6 and 7. In Figure 5, the *shapes* of the tile motifs are varied via a vertical gradient in the Bézier control parameter u. However, in this case the choice of color scheme (black on white or white on black) for each tile is made *randomly*, so that the average brightness remains uniform. By spatially varying u in two dimensions, more complex images can be constructed with uniform average brightness using this approach with either duotone or contour style tiles.

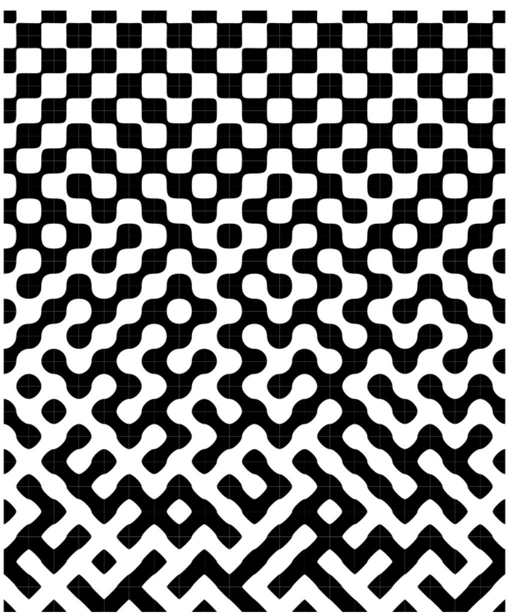

Figure 5: *Tile shape gradient with uniform average brightness.*

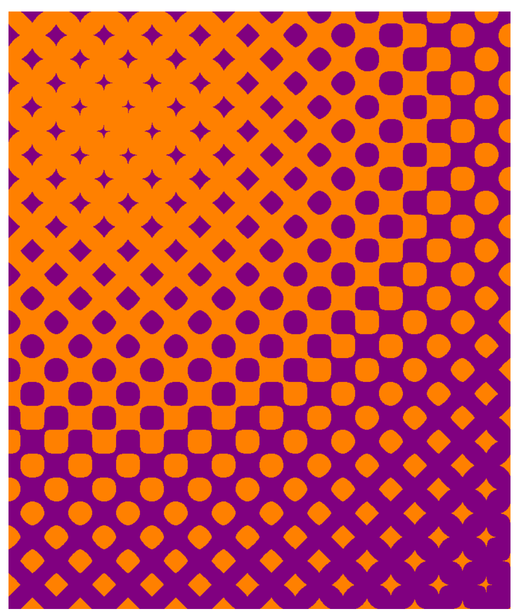

Figure 6: *Radial brightness gradient.*

In creating a mosaic with brightness gradients, the edge color-matching rule for neighboring tiles results in dots of contrasting color at the corners where four tiles come together. For the lowest and highest values of brightness, the dots take the form of four-pointed stars. As brightness progresses from either extreme toward the halfway point, the shape of the contrasting dots evolve as they grow, first to a diamond shape, then to circles and for brightness near 0.5 the dots are nearly square, and larger connected regions may form. An

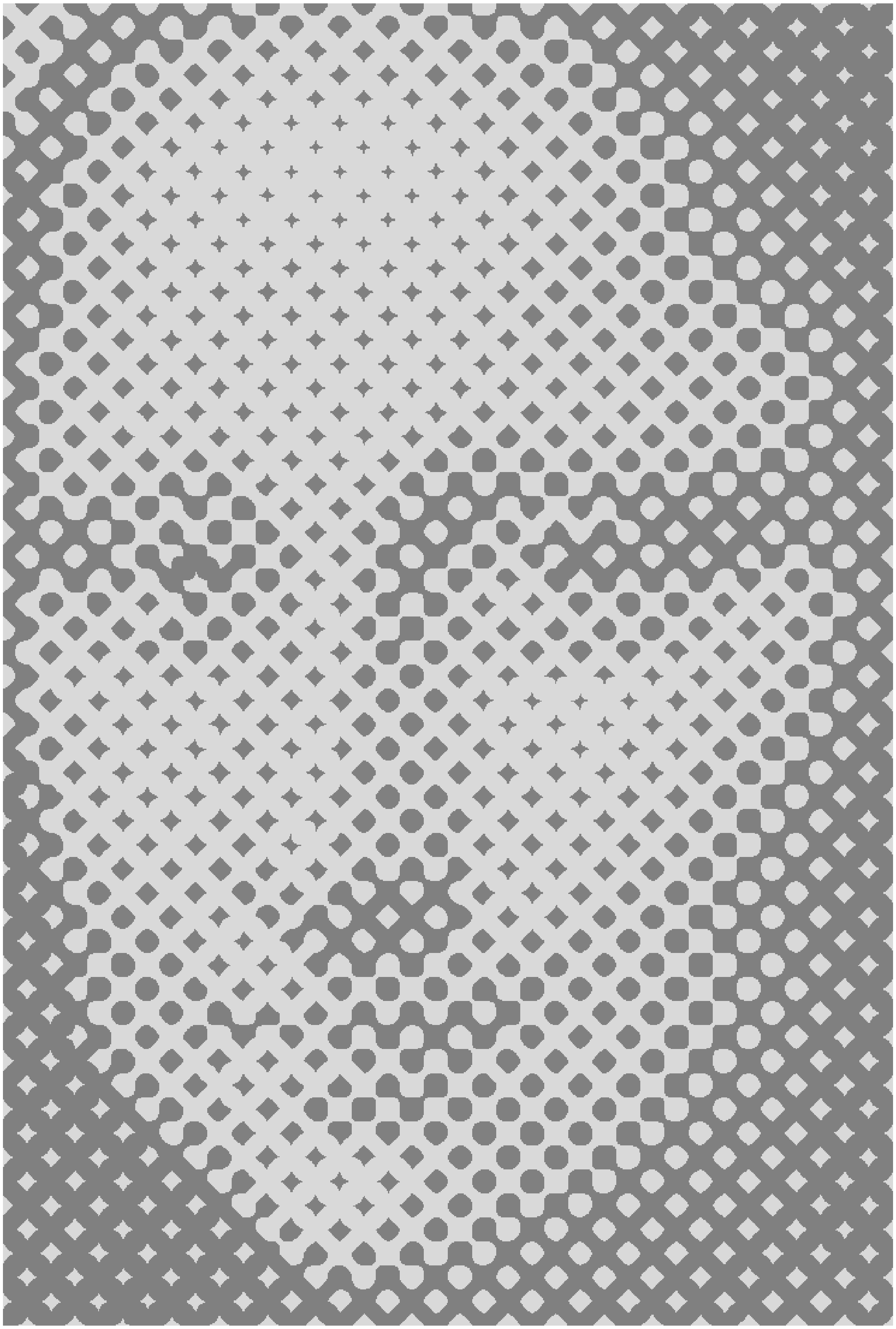

Figure 7: *The system for controlling brightness by using Bézier curves as color boundaries at tile corners, applied to produce a figurative mosaic.*

example mosaic of a simple graphic with a gentle radial color gradient is shown in Figure 6. Figure 7 shows a mosaic based on an image with greater detail, a cropped version of the Mona Lisa [20].

Tiles with Adjustable Brightness for Figurative Mosaics: Considerations for Fiber Media

Depicting figurative mosaics in textiles necessitates a balance between image resolution and complexity, and as in any medium, achieving an aesthetic balance is central to the artistry involved. In the case of quilts, the time-honored method of assembling individual square quilt blocks to create the full mosaic could be readily applied. Each quilt block could be constructed as an individual tile fabricated with brightness appropriate for its location in the completed quilt. Additional patchwork and quilting constraints include the size of the individual blocks and the size of the completed quilt. Estimating a reasonable quilt block minimum of 4 inches by 4 inches, the mosaic of the Mona Lisa shown in Figure 7, at 37 tiles by 56 tiles, would result in a quilt nearly 4 meters by 6 meters, pushing the limits of both quilt construction and display, but at 21 by 27 tiles, the graphic in Figure 6 would be in the size range of conventional quilts.

Practical constraints also apply in the creation of knitted images. A programmable knitting machine offers advantages over hand knitting for producing images, including ease of implementation, knitting speed and potentially higher image resolution (up to $\sim$ 10 stitches per inch may be possible). On the other hand, color options in the fine gauge "lace weight" yarn cones as needed for high resolution machine knitting are more limited than those for hand knitting yarns, so using two yarn colors is an attractive method for machine knitting color gradients. Another constraint with knitting is the need to limit the length of "floats," the connecting strand of the yarn color not in use across rows on the backside of the piece. The described tiling approach to constructing duotone gradients addresses this constraint because even the brightest and darkest tiles have at least one contrasting stitch in two corners.

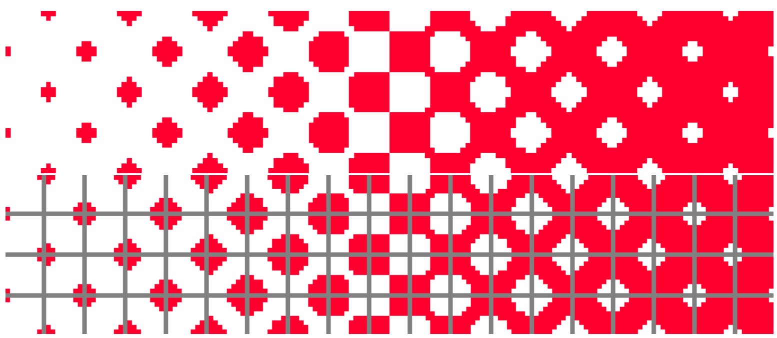

Figure 8: *The adjustable brightness tile system adapted for discrete/low resolution grid based graphics media, such as knitting. Each tile is "rasterized" to a 8×8 grid of square stitches, with gray "grout" lines added in the lower part of the figure to highlight the individual square tiles.*

In adapting tile mosaics for knitting (or needlepoint, cross stitch, etc.), the discrete size of individual stitches must be accounted for. Figure 8 is a rendering of a horizontal brightness gradient in which each tile is formed by an 8×8 grid of stitches. At 8 stitches per tile and estimating 8 stitches per inch, a knitted version of the Mona Lisa example would be under 1 meter by 2 meters, suitable for a blanket or wall hanging.

Concluding Remarks

A method using Bézier curves for smooth and continuous control of the curvature of the light/dark boundary in square tiles with contrasting corners has been introduced as a practical framework for programming tile brightness over a wide dynamic range for the creation of figurative mosaics. The method is readily adapted to a variety of creative media and has been described in sufficient detail for replication by others. Just a few examples of its application have been introduced in this article and numerous extensions remain unexplored. Crafting opportunities abound!

References

[1] R. Bosch. *Opt Art*. Princeton University Press, 2019.

[2] R. Bosch and U. Colley. "Figurative mosaics from flexible Truchet tiles." *Journal of Mathematics and the Arts*, vol. 7, no. 3-4, 2013, pp. 122–135.

[3] B. Brackman. *Encyclopedia of Pieced Quilt Patterns*. Paducah, KY USA: American Quilter's Society, 1993.

[4] C. Browne. "Truchet Curves and Surfaces." *Computers & Graphics*, vol. 32, no. 2, 2008, pp. 268–281.

[5] C. Carlson, "Multi-Scale Truchet Patterns." *Bridges Conference Proceedings*, Stockholm, Sweden, July 25–29, 2018, pp. 39–44, http://archive.bridgesmathart.org/2018/bridges2018-39.pdf.

[6] B. Casselman. From Bézier to Bernstein. 2008. https://www.ams.org/publicoutreach/feature-column/fcarc-bezier.

[7] A. H. Christie. *Pattern Design: An Introduction to the Study of Formal Ornament*, 2nd ed. Dover, 2011. ch. 10. pp. 271–310.

[8] R. T. Farouki. "The Bernstein polynomial basis: A centennial retrospective." *Computer Aided Geometric Design*, vol. 29, 2012, pp. 379–419.

[9] M. Goldapp. "Approximation of circular arcs by cubic polynomials." *Computer Aided Geometric Design*, vol. 8, 1991, pp. 227–238.

[10] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*. New York: W. H. Freeman and Co., 1986.

[11] G. Kirdeikis. Truchet Tiles (Rhino Grasshopper Tutorial). March 2019. https://youtu.be/DIc7a2mectY.

[12] A. Kreiner, "Checkerboard Quadrilateral Mosaics." *Bridges Conference Proceedings*, (virtual), Aug. 1–5, 2020, pp. 207–214. http://archive.bridgesmathart.org/2020/bridges2020-207.html.

[13] K. Mitchell, "Generalizations of Truchet Tiles." *Bridges Conference Proceedings*, (virtual), Aug. 1–5, 2020, pp. 191–198. http://archive.bridgesmathart.org/2020/bridges2020-191.html.

[14] S. Mortenson. Approximate a circle with cubic Bézier curves. https://spencermortensen.com/articles/bezier-circle/.

[15] D. A. Reimann, "Patterns from Archimedean Tilings Using Generalized Truchet Tiles Decorated with Simple Bézier Curves." *Bridges Conference Proceedings*, Pécs, Hungary, July 24–28, 2010, pp. 427–430, http://archive.bridgesmathart.org/2010/bridges2010-427.html.

[16] R. Sarhangi, S. Jablan, and R. Sazdanovic, "Modularity in Medieval Persian Mosaics: Textual, Empirical, Analytical, and Theoretical Considerations." *Bridges Conference Proceedings*, Winfield, Kansas USA, July 30–Aug. 1, 2004, pp. 281–292, http://archive.bridgesmathart.org/2004/bridges2004-281.html.

[17] M. D. Shepherd. "Variations on Snake Trail Quilting Patterns." *Figuring Fibers*. C. Yackel and sara-marie belcastro, Eds. American Mathematical Society, 2018. ch. 4.

[18] C. S. Smith and P. Boucher. "The Tiling Patterns of Sébastien Truchet and the Topology of Structural Hierarchy." *Leonardo*, vol. 20, no. 4, 1987, pp. 373–385.

[19] S. Truchet. "Memoire sur les Combinaisons." *Histoire de l'Académie Royale des Sciences*, 1704, pp. 363–373 and plates 12–20.

[20] Wikipedia (Public Domain). Mona Lisa. https://en.wikipedia.org/wiki/Mona_Lisa.