Tensegrity Polyhedra Models

Mircea Draghicescu

Portland, Oregon, USA; mircea@itsphun.com

Abstract

We demonstrate two simple approaches for building tensegrity models of arbitrary polyhedra. Participants will have the opportunity to build their own models. The methods described here can be used for mathematical art projects or as a teaching aid in a 3D geometry class.

Introduction

A tensegrity construct is a stable 3D structure composed of isolated components (usually bars or struts) under compression connected by cables or strings under tension. Tensegrity was popularized starting in the late 1940s by Richard Buckminster Fuller (who coined the term "tensegrity", short for tensional integrity) and his student, the artist Kenneth Snelson. Since then, tensegrity principles have been used to create spectacular art installations (e.g., Figure 1) and have found many applications in architecture and medicine (see, for example, [5], [6], [7]).

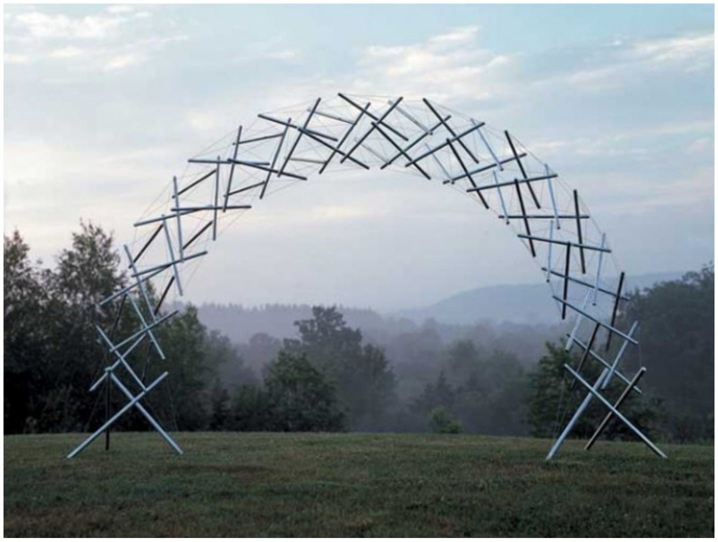
For an amateur mathematical artist, a very attractive property of tensegrity is the endless variety of objects that can be strung together to form a decorative object (see Figure 2). Building a tensegrity model, however, is not easy. Unlike other modular geometric constructions that can be assembled by adding pieces, one at a time, to a growing partial structure, a tensegrity construct does not take shape until the last bar is connected and all the elements are tensioned. The building process requires a system to adjust the length of the strings and some scaffolding to keep the partial model in place during construction.

A well-known shortcut uses elastic strings (e.g., rubber bands) to create the necessary tension, see Figures 3 (copied from [4]) and 4 (b). We illustrate a variant of this method that uses custom bars joined with pieces of thin bungee cord, see Figure 6. In our approach the strings always run between the end of one bar and the middle of another one and each bar middle is connected to exactly two ends; the same idea is used, for example, in Figure 4 (a). This restricted form of tensegrity is very well suited to building topological models of polyhedra as defined in the next section. Interested readers are referred to [1], [2], and [3] for more details.

We will use the same connection pattern to demonstrate a second tensegrity construction shortcut which consists of replacing the rigid bars with flexible ones. Unlike in the previous approach, the necessary tension is produced by the deformed bars themselves. The advantage of this method is that the bars can be connected with fixed length strings; the bars will flex independently allowing a model to settle into a stable shape. The construction process is a bit more complicated, but produces nicer looking models, see Figure 7.

Traditionally, tensegrity strings only connect to bar ends; we can always convert our models to satisfy this requirement by replacing a string to a bar middle by two equal-length strings to the bar ends. The shorter string to the middle simplifies the construction process and produces models that appear visually less crowded; compare, for example, Figures 4 (b) and 7 (b).

In the workshop we will build several tensegrity models and talk about their relation to the modeled polyhedra. The emphasis here is on simplicity, ease of use and reuse, and the application of these techniques in an art or math class, but depending on the participants interest, we can also discuss other methods for creating tensegrity works of more artistic value.



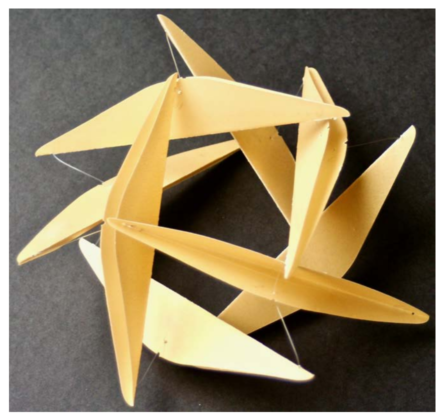
(a) *Rainbow Arch*

(b) *Needle Tower*

Figure 1: *Tensegrity sculptures, Kenneth Snelson*

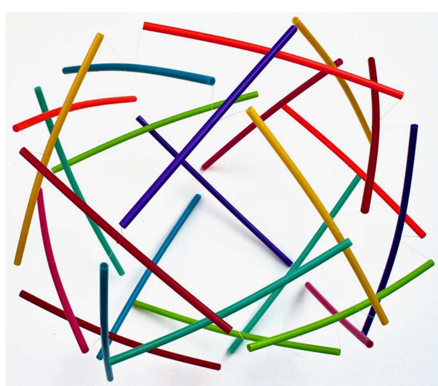
(a) *Pencils*

(b) *Driftwood*



(c) *Paper*

(d) *Wood dowels*



(e) *Drinking straws*

(f) *Paper*

Figure 2: *Tensegrity models built by the author using various materials*

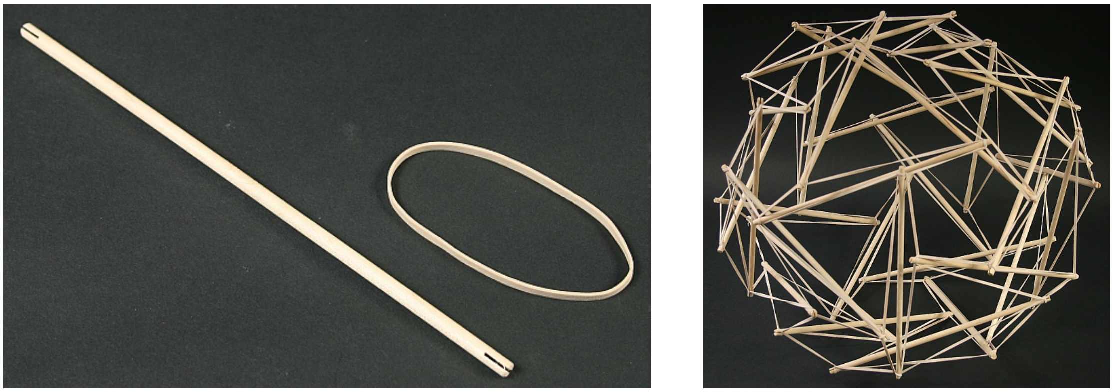
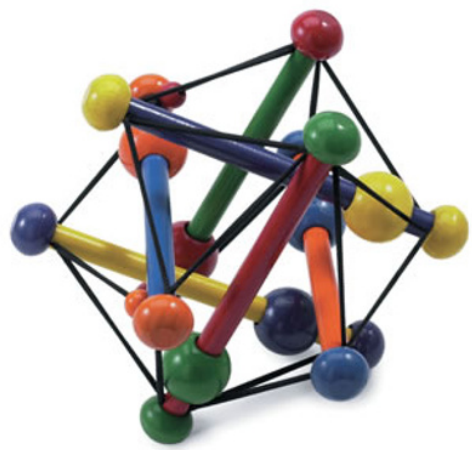


Figure 3: *Rubber band tensegrity by George Hart*

(a) *Floating Logs*



(b) *Skwish*

Figure 4: *Tensegrity creations, Tom Flemons*

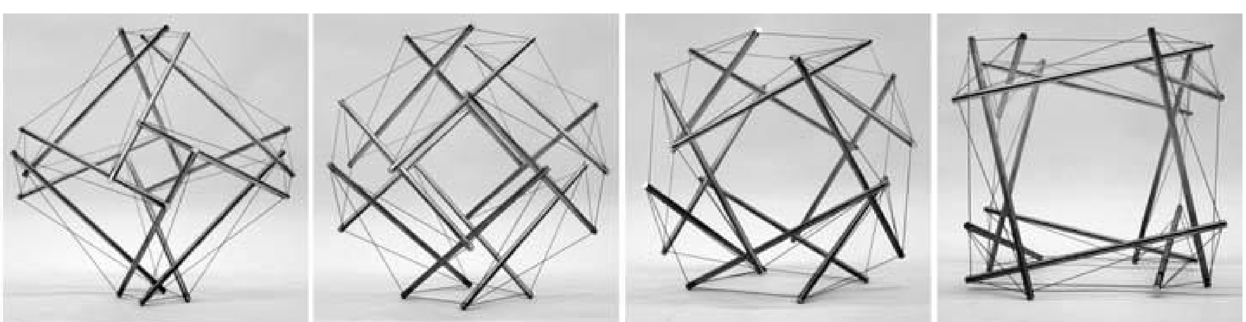
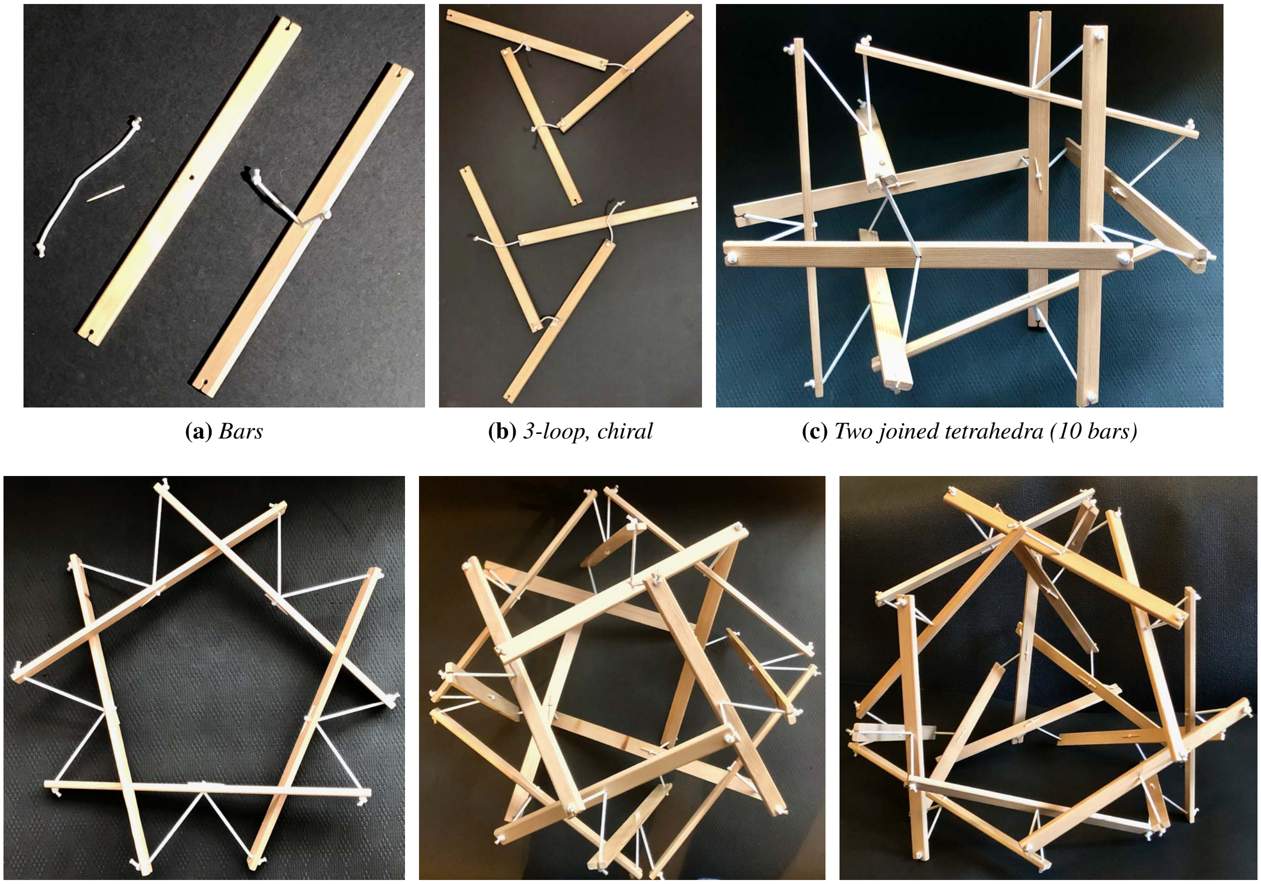


Figure 5: Cube Metamorphosis*, Kenneth Snelson*



(a) *Bars* **(b)** *3-loop, chiral* **(c)** *Two joined tetrahedra (10 bars)*

(d) *Pentagonal hosohedron/dihedron* **(e)** *Square antiprism* **(f)** *Truncated tetrahedron*

Figure 6: *Elastic cord tensegrity. To estimate the size of the models consider that each bar is one foot long.*

Tensegrity and Polyhedra Models

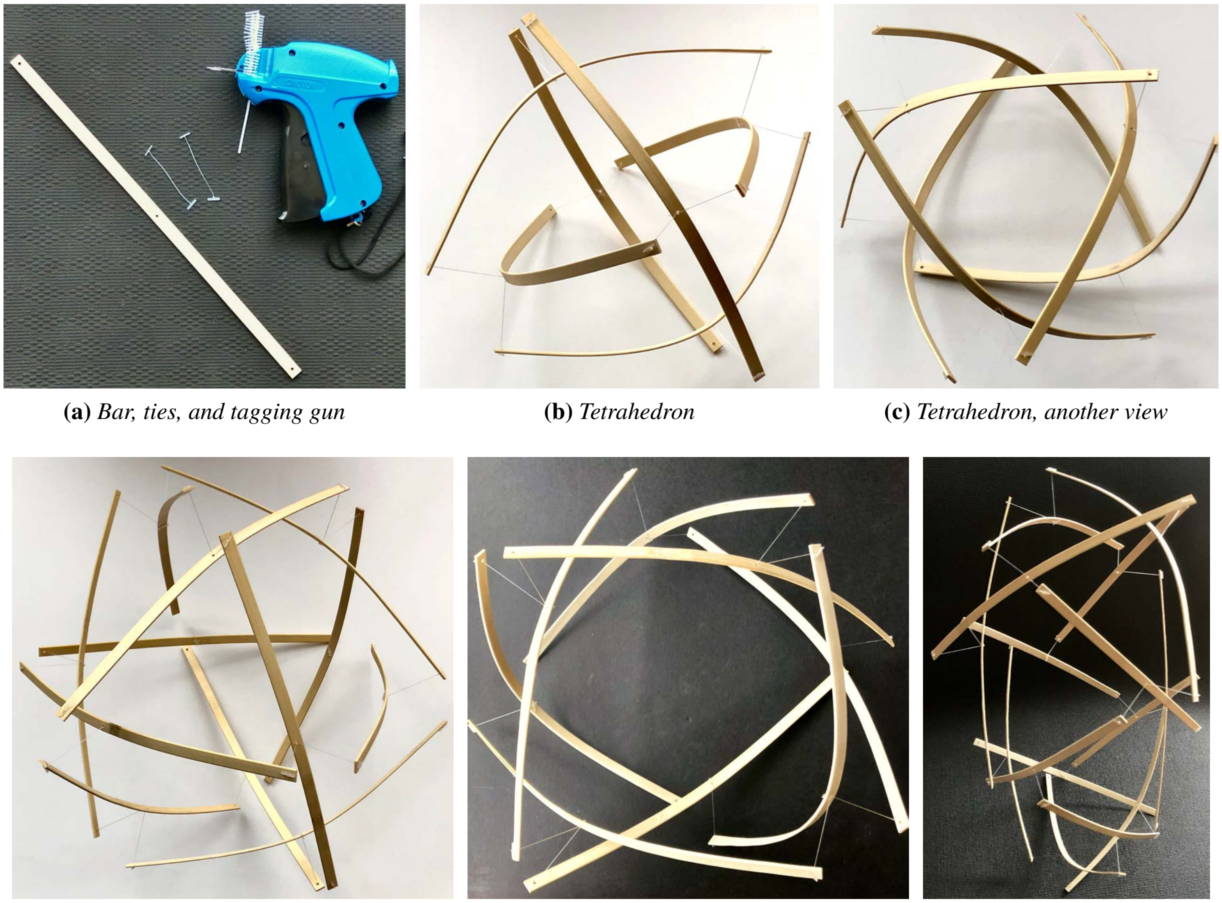
As mentioned in the Introduction, we use a restricted form of tensegrity where a bar end is connected only to another bar middle and each bar middle is connected to exactly two ends. Each bar is thus connected to four other bars: two at its ends and two to its middle.

Definition

We will consider here polyhedra topologically equivalent to a sphere; they can be either flat-faced or spherical. The type of connection described above allows us to build a model of an arbitrary polyhedron in the following sense:

1. there is a one-to-one correspondence between the model bars and the polyhedron edges, and
2. two model bars are connected iff the corresponding edges intersect and are on the same face, i.e., the edges share both a face and a vertex.

Note that two edges can be connected more than once if they share more than one vertex or more than one face (the connectivity graph of the edges is, in general, a multigraph). This can happen for some spherical polyhedra, see Figure 6 (d).



(a) *Bar, ties, and tagging gun* **(b)** *Tetrahedron* **(c)** *Tetrahedron, another view*

(d) *Cube/octahedron* **(e)** *Square pyramid* **(f)** *A stack of two triangular prisms*

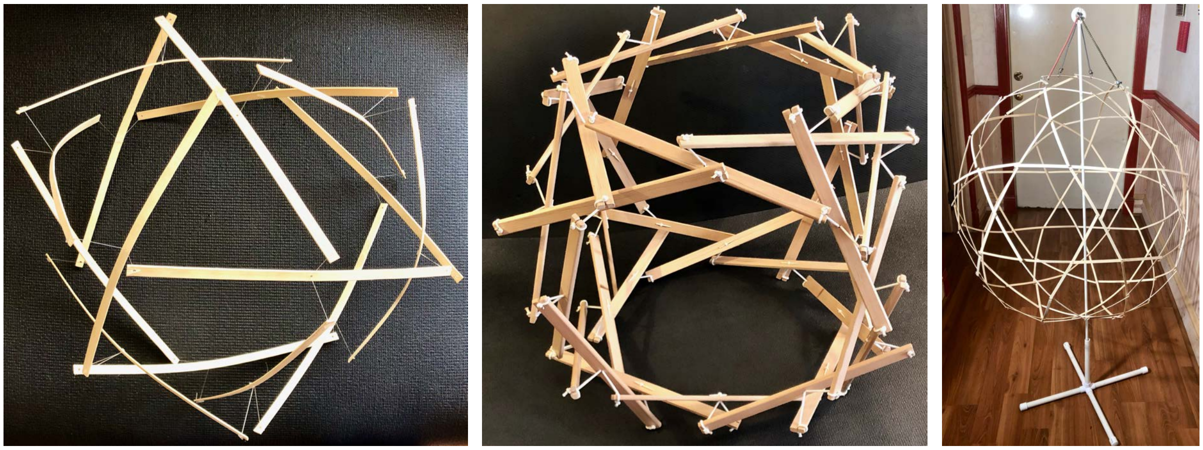
Figure 7: *Flexible bar tensegrity*

Duality

With this definition of edge connectivity, the connectivity graphs of the edges of dual polyhedra are isomorphic, therefore dual polyhedra generate the same model. A model $M(P)$ of a polyhedron P usually resembles both P and its dual P', but it might also settle into an unexpected shape that does not resemble either.

A loop of n bars in $M(P)$ (see, for example, the "triangle" in Figure 6 (b)) corresponds to an n-sided face of either P or P' (or, same thing, to a vertex of degree n of either P or P'). Another interpretation is that such a loop corresponds to an n-sided face of the *rectification* of P. In Conway's polyhedron notation, the rectification of P is denoted by aP (operator a, or *ambo*, applied to P); note that $aP = aP'$. For example, the loops of the tetrahedron model $M(T)$ in Figure 7 (b,c) correspond to the faces of the ambo tetrahedron aT, which is the octahedron, while the loops in $M(C) = M(O)$ (Figures 2 (d) and 7 (d)) correspond to the faces of $aC = aO$, the cuboctahedron.

We can make $M(P)$ visually closer to P by enlarging the loops corresponding to the faces of P and shrinking the loops corresponding to the vertices of P (doing the opposite will bring $M(P)$ closer to P'). To do this we split the connection point in the bar middle into two connection points, each attached to a single string, and move these points closer to opposite ends of the bar. The enlarging/shrinking of a loop can then be achieved by choosing to which point to connect the strings that form the loop. When the string to a bar middle



(a) *Triangular cupola* **(b)** *Truncated cube* **(c)** *Truncated icosahedron*

Figure 8: *More models*

is replaced by two strings to the bar ends, as described in the Introduction, the same effect can be achieved by varying the lengths of the two strings. Figure 5 offers a beautiful illustration of this transformation: a model that is closer to an octahedron (left) is gradually changed to one that is closer to a cube.

Chirality

Model loops with three or more bars are chiral; Figure 6 (b) shows the two orientations of the 3-loop, viewed from outside a model. The $M(P)$ loops that correspond to the faces of P have one orientation, while the loops that correspond to the faces of P' have the opposite orientation. $M(P)$ is chiral if P is chiral or if $P \neq P'$ and either P or P' has a face with three or more sides.

Symmetry

The symmetry of a polyhedron P is, in general, different from the symmetry of its model $M(P)$. For example, the tetrahedron model (Figure 7 (b)) has pyritohedral, not tetrahedral, symmetry. $M(P)$ does not have any of the reflection symmetries of P, but, if P is self-dual (e.g., a pyramid), it has some new reflection symmetries which are not present in P. For example, the model of the square pyramid in Figures 2 (c) and 7 (e) has rotoreflection symmetries with 45° rotations about the axis through the middle of the square loops.

Other properties

The models built in this manner have two interesting properties, which will be demonstrated in the workshop:

- In general, a model can be extended by adding an additional bar. This extension corresponds to splitting a polyhedral face into two faces by adding a diagonal, which is possible whenever the modeled polyhedron, or its dual, has a face with at least four sides. The only flat-faced polyhedron that does not satisfy this requirement is the tetrahedron: we cannot add a new bar, for example, to the model in Figure 7 (b). (In fact, the number of edges in a "proper" polyhedron is either 6 or at least 8.)
- Any two models can be joined into a larger model by first removing one bar from each and then tying the loose ends. Indeed, removing a bar and the two strings attached to its middle leaves two loose strings attached to two other bars' ends and two loose bar ends without an attached string. We can then join the two models by attaching the loose strings from one to the loose bar ends from the other. The corresponding topological transformation on polyhedra has no geometric interpretation.

Joining in this manner two proper polyhedra might create an improper one, for example joining two tetrahedra will form a self-dual spherical polyhedron with four triangular and two quadrilateral faces whose model is shown in Figure 6 (c) (a tetragonal hosohedron where two opposite lunes are each split into two triangles). Model composition offers many interesting possibilities of building new models, especially when the tensegrity system is used as a construction toy.

Construction Details

Rigid Bars with Elastic Cords

For the first tensegrity construction method we use 1/4”×3/4” wood moulding available at a lumber or hardware store. To make the tensegrity bars, we cut the moulding strip into 12” pieces and drilled three 5/32” holes, one in the center and two close to the ends of the bar. The bars are connected with pieces of 1/8” bungee cord which are folded in half, inserted through the center hole of a bar and held with a small object such as a toothpick or paper clip (see Figure 6 (a)). We also cut short notches at the end of the bars, connected to the two end holes. The notches make it easy to attach the bungee cords during model construction (and also detach them, if needed). The result is a tensegrity construction system that is easy to use, produces sturdy models, can be reused repeatedly, and is thus well suited to a classroom environment.

Flexible Bars with Tagging Ties

To illustrate the second building method, we use 3/8”×12” craft bamboo strips available, for example, from several Amazon vendors. The preparation of each piece consists again in drilling three holes, one 5/64” in the center and two 3/32” at the ends. (The center hole should be as small as possible since it weakens the strip at the point of maximum stress; there are also more elaborate connection solutions than do not involve drilling this hole.) To connect the strips we use plastic tagging ties that can be attached either by hand or with a tagging gun, see Figure 7 (a). The tagging ties do not stretch; different models might require different length ties to obtain the desired visual effect and avoid bending the bars beyond the breaking point (we use one, two, or three inch long ties). In general, large, symmetric models that approximate a sphere require shorter ties; in Figure 8 (c), for example, we folded the one inch ties in half. At the limit, the bars are simply tied together and, while the model is still under tension, it is debatable if it is still “tensegrity”. The bamboo strips are much more fragile than the wood bars, but they can still be reused by cutting the plastic ties.

Tensegrity Construction Tips

The shortcuts presented here do not produce “true” tensegrity constructs, i.e. ones that have neither elastic strings nor flexible rods. Such structures, even small art projects, are harder to build; here are a few tips.

Carefully choose your materials. Estimate the tension needed and pick the bars and strings accordingly. Weight matters: heavy bars might be stronger but will also require more tension to keep the model from sagging under its own weight. Bars that have a little give are easier to work with than absolutely rigid ones.

Pick a method of tying the strings that allow some length adjustment. It is usually hard to compute exactly the length of the connecting strings and some late adjustments seem to be always needed. (Sometimes this issue can be bypassed: [3] describes an infinite class of polyhedra whose tensegrity models use a single loop of string that connects all the bars in the model, thus making tension adjustment very easy, see Figure 2 (c),(e),(f).)

Rubber bands are still useful. A method that we often use consists in building the entire model using rubber bands, then cutting and replacing the rubber bands, one by one, with the real strings. This works because removing just one rubber band will not collapse the entire model. The rubber bands can be attached to the bars with some simple knots.

Use gravity to help during construction. This is a very useful tactic, especially for larger polyhedra models: start with a loop, suspend it so that it stays horizontal, and keep adding bars to the structure, from top to bottom, one circular layer at a time (see Figure 8 (c)). This method brings order to the building process and keeps the partially build model from turning into a jumble of strings and bars.

The shape of a model can be changed. Once all connections are made, the models built with the materials described here will settle into a (sometimes unexpected) shape over which we have no control. When building an individual model, we can adjust its shape by choosing bars and/or connecting strings of different length, or by moving a bar's connection point away from the center.

Classroom Use

With minimal resources and some simple preparation, the methods described here can be used to make a tensegrity construction system that can be used in a classroom. Tensegrity polyhedra models can be used in a math class to sharpen the students' 3D visualization skills and introduce many geometrical and topological concepts related to polyhedra. To build a model of a given polyhedron, a student must consider both the polyhedron and its dual; since the model might not resemble either, the construction process requires some thinking and planning ahead. Comparing and contrasting the symmetry of a polyhedron with the symmetry of its model offers a good opportunity to learn about symmetry types.

Conclusion

We present two simple methods of building tensegrity structures that can be used for creating interesting artistic structures or as teaching aids in a math class. In the workshop we illustrate these methods using specially prepared construction elements (bars), but the methods can be easily adapted to other materials, shapes, and connection mechanisms.

References

[1] M. Draghicescu. "A General Method for Building Topological Models of Polyhedra". *Bridges Conference Proceedings*. Waterloo, Ontario, Canada, July 27–31, 2017, pp. 175–182. http://archive.bridgesmathart.org/2017/bridges2017-175.pdf.

[2] M. Draghicescu. "Building Polyhedra Models for Mathematical Art Projects and Teaching Geometry". *Bridges Conference Proceedings*. Linz, Austria, July 16–20, 2019, pp. 629–634. http://archive.bridgesmathart.org/2019/bridges2019-629.pdf.

[3] M. Draghicescu. "Single-threaded Polyhedra Models". *Bridges Conference Proceedings*. Online, Aug. 1–5, 2020, pp. 281–288. http://archive.bridgesmathart.org/2020/bridges2020-281.html.

[4] G. Hart. "Math Monday: Tensegrity Balls". *Make: Online* (Oct. 2011). https://makezine.com/2011/10/24/math-monday-tensegrity-balls.

[5] E. Heartney and K. Snelson. *Kenneth Snelson: Forces Made Visible*. Hard Press Editions, 2009. https://books.google.com/books?id=E0%5C_cQC0z5HEC.

[6] R. Motro. *Tensegrity: Structural Systems for the Future*. Elsevier Science, 2003. https://books.google.com/books?id=0n0K6-zOB0sC.

[7] A. Pugh. *An Introduction to Tensegrity*. Dome series. University of California Press, 1976. https://books.google.com/books?id=McEOfJu3NQAC.