Exploring Music as a Geometric Object

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Abstract

In this manuscript, we strive to define a scheme for mapping music into a three dimensional structure. To achieve our goal, we convert choruses of Beatles songs into curves. The manifestation of these geometric objects lets us observe the songs' complexity, as well as analyze it using mathematical tools from differential geometry. Our aim is that the viewer will be able to identify the song and its composer by observing the geometric object (curve) alone. We believe that these ideas can offer composers a new perspective on their own music, as well as offer non-professional audiences a glimpse into the complexity of composing.

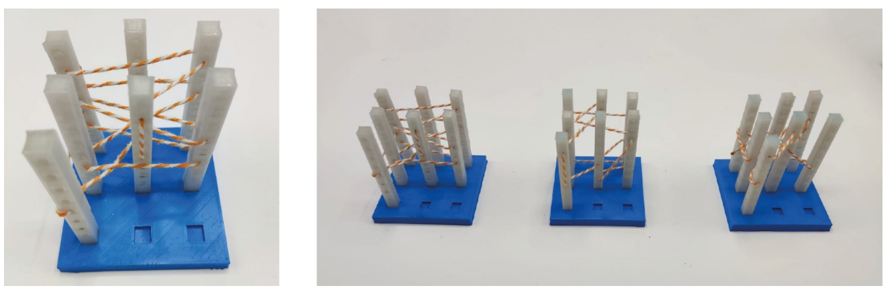

Figure 1: *Songs which have been transformed into* 3*-dimensional physical objects. The poles represent the chords as coordinates. The wire moves along the coordinates to construct a curve of a given song.*

Introduction

Our starting point for this research was a friendly discussion of amateur musicians about the vibration that occurs in low frequencies, which gives a kind of physical feel to music. It led us to think about how we might be able to define music as a physical object and how, by observing the music as an object, we could understand its complexity and its degree of similarity to other given songs.

In this work, we try to offer a new point of view about the connection between mathematics and music in the special case of the western chromatic scale.

Our method maps a given song, which is constructed of triads (basic chords in classical western harmony), to curves in three dimensions, which can be explore from a mathematical point of view.

The western chromatic scale is based on 12 notes :

$$\{A, A\#, B, C, C\#, D, D\#, E, F, F\#, G, G\#\} ,$$

each of which can be represented by a number in the range of 0 through 11. Chords are defined as a set of notes played together. A triad is defined as a set of 3 different notes as $F_{maj} = (F, A, C) = (8, 0, 3)$. By this definition, if we transform the notes to natural numbers then the notes can be represented as elements of $\mathbb{Z}_{12}$ and a given triad is defined as elements of $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. Note that our identification does not distinguish different voicings of a given chord. The major triad with root r, which is the relative shift in terms of semi-tones, is composed of three tones, $(r, r+4, r+7)$. As r represent the same tone as $r+12$, the respective intervals within the triad are $(4, 3, 5)$, with the third entry, 5, being the distance from $r+7$ to $r+12$. Similarly, the minor triad with root r is composed of the three tones $(r, r+3, r+7)$, inducing the intervals $(3, 4, 5)$.

In [1, 3] a mathematical approach based on ideas in group theory has been introduced to explore music and harmony improvisation methods.

In this collaboration of amateur musicians, we wonder how we might give a visual representation to a set of triads. Can we find a way to view music and represent the song in a similar way to reading notes from a music sheet? Can we define and explore music using geometric properties?

Our approach is restricted to sequences of triads, where the triads are points in three dimensional space. The progression along the song defines a directed edge which is determined by every two successive triads; and all of which define a curve. This curve signifies the complexity of the song or chorus and serves as the basis for comparison between two different songs or choruses, all using geometric properties.

After brainstorming as a team, we decided to explore this geometric object by using tools from differential geometry such as total curvature which will be explained in the Preliminaries. We believe that by using this method we can compare songs and give a non-professional audience an easy glimpse into the relative complexity of songs without requiring any prerequisite understanding of music theory.

Preliminaries

In this section we will provide the essential mathematical definitions and concepts we intend to use to explore music.

In differential geometry, curvature is the amount by which a curve deviates from being a straight line. In the plane, if a curve γ is represented by a parametric representation $\gamma(t) = ((x(t), y(t)))$, then the curvature κ of a given point can be calculated by

$$\kappa = \frac{\det(\gamma', \gamma'')}{||\gamma'||^3} .$$

This definition can be generalized to any number of dimensions.

In the plane, the sign of the curvature (positive or negative) is determined by the direction of the curve: clockwise gives negative curvature while counterclockwise gives positive curvature. In other dimensions we will take the absolute value. The curvature of a line is zero, and the curvature of a circle with radius R is constant $\kappa = \pm\frac{1}{R}$, since it deformed uniformly.

This leads us to the definition of total curvature for a given curve (see [4]), which is the summation of curvature at all points along the curve. For example, in a triangle, there are only 3 points of curvature, which are defined by the vertices. In the plane, if the curve is closed and simple (does not intersect itself), then the total curvature is $\pm 2\pi$, so we can conclude that the total curvature of a triangle is 2π or -2π, depending on the direction in which the curve has been defined (see Figure 2).

If the curve is closed but not simple, the approach is to calculate it with respect to total curvature based on splitting the given curve into a collection of simple closed curves-clockwise or counterclockwise, each of which define a total curvature of $\pm 2\pi$, and the sum of this closed simple curves determine the desired total curvature of the initial closed curve. This number which is defined by the number of laps (+1 if the lap is counterclockwise and -1 clockwise) is called the curve index, which is a topological property (for more details see [4]).

Our Results

Since we would like to represent the music in three dimensions, we will constrain our chords to triads. If a chord is defined by more than three notes (see Figure 2), we choose the three first notes as has been done in [1]. Each triad is represented as a point in three dimensional space, and each sequence of points creates a curve which is constructed of straight lines.

In order to convert a computer plot into a physical object, we first created a "working model" or a technical model (see Figure 1). Through these models, we determine that the main elements we wish to explore visually are (1) the curve and (2) the coordinate system. The chorus, defined by a sequence of triads, is represented by a thick simple wire, which is a convenient way to build a model in three dimensional space with real materials and convey complexity, as in Figure 1.

For the sake of simplicity, we focused on choruses of songs in which the song starts and ends in the same triad, to produce a closed curve with a given index. In the case of two-sided direction edges, we will define the curvature as zero. Note that the absolute value of the curvature is well-defined in three dimensions, but the total curvature of a closed curve is meaningless. This leads us to project the three dimensional curve onto the plane in three different ways (we will omit one of the axes in each of the projections), where the total curvature (in each case) is well-defined and can be classified as an index of a closed curve. This projection of choruses onto the plane leads to three total curvatures, which, in our geometric approach will be the songs' respective characterizations, i.e., we will sort songs by the ordered total curvature projections.

Figure 2 describes the process for exploring the geometric object of the song (chorus) “Get Back”, by the Beatles. From left to right, given a sequence of triads which represent the chorus, a three dimensional oriented curve (second image) is defined – in this case, creating a triangle. To explore this three dimensional triangle, we project it into the plane. The (x, y) projection (green triangle) is a closed counterclockwise curve with a respective total curvature of 2π. The second projection leads to the red triangle, with a respective total curvature of -2π (closed clockwise curve). The third projection leads to the blue triangle, with a total curvature -2π. So, using our visualization, all choruses (geometric objects) which are defined by $(2\pi, -2\pi, -2\pi)$ as a song (chorus) have the same geometric properties. Comparing Figures 2, 3, 4 and 5 reveals the respective complexity projection and total curvature projection of each chorus. Lastly, two choruses are equivalent if the respective total curvature projections are equal.

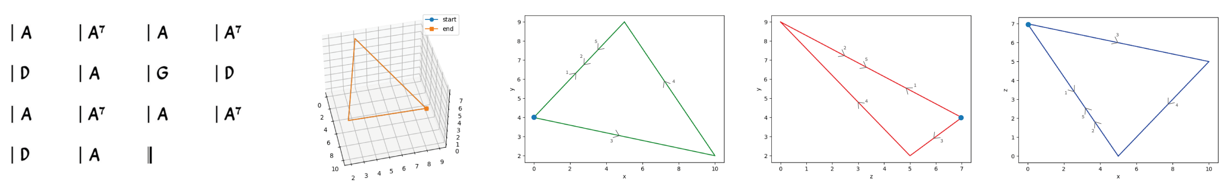

Figure 2: *The Beatles: Get Back. From left to right: the chorus, the three dimensional representation and the three projections. The dot in every curve is the first triad in the chorus. The total curvature of the respective projection in the plane is* $(2\pi, -2\pi, -2\pi)$.

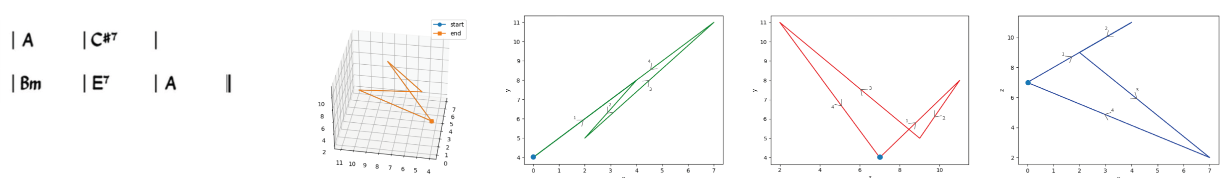

Figure 3: *The Beatles: Like Dreamers Do. The respective projections total curvature is* $(2\pi, 0, -2\pi)$.

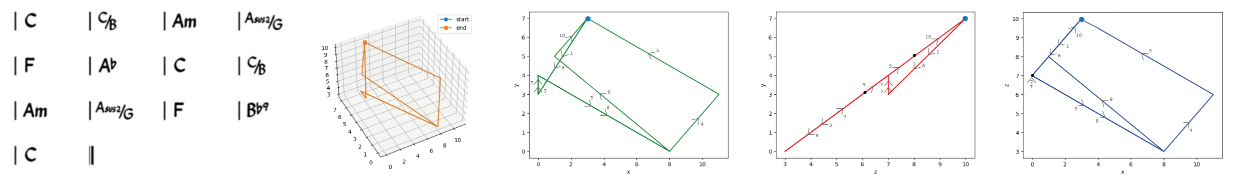

Figure 4: *The Beatles: Hello, Goodbye. The respective projections total curvature is* $(0, -4\pi, 4\pi)$.

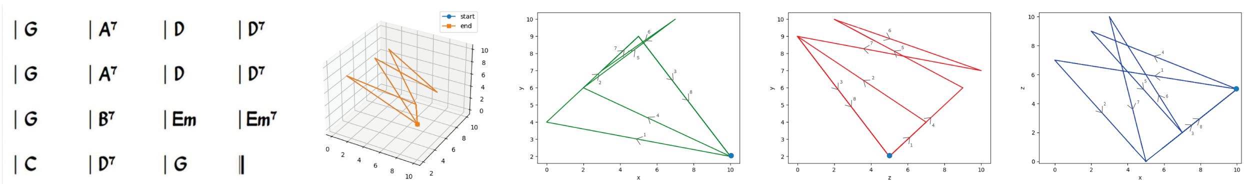

Figure 5: *The Beatles: All You Need Is Love. The curve reveals the complexity of this song. The respective projections total curvature is* $(-4\pi, 2\pi, 6\pi)$.

Future Work

From a design point of view, we are working on models which the viewer can explore physically. From a mathematical point of view, our intention is to define curves with respect to a complete song (chorus, verse, c part). Each curve will be identified by a different color which, combined, will represent the song (and further, explore it with similar tools to the chorus geometric exploration). In addition, we intend to explore chords consisting of essentially the same notes but distinguished by the root. Thereafter, we intend to generalize our result to surfaces, in a similar way to [2], using the Gauss–Bonnet Theorem (see [4]), which classifies topological surfaces by the total Gaussian curvature and geodesic curvature.

Summary and Conclusions

We show how the chorus of a song can be represented as a curve and offer an approach to exploring it using geometric ideas. Using these approaches can convey the complexity of a given song to a non-professional musical audience. It also offers a new geometrical perspective through which musicians can characterize their own compositions. We believe that our model can connect compositions from different musical styles using geometric objects or geometric properties such as total curvature.

References

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