Ideal Triangulations of Surfaces: A Comic

Alan McLeay

Instituto de Ciencas Matimáticas, Madrid, Spain

Abstract

This short paper discusses the mathematical comic strip *Ideal Triangulations of Surfaces* created by the author [1]. It shines light on some of the motivations behind its creation, and also some interesting aspects which could lend themselves to outreach and teaching.

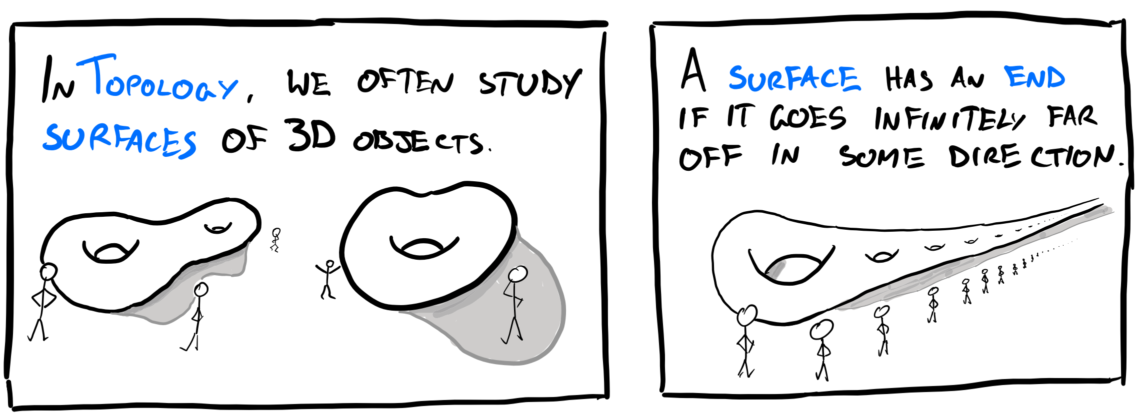
A Comical Proof

Many of the ideas, techniques, and results that we see in low dimensional topology have accompanying pictures which, albeit informally, convey the meaning behind the notation we use. This allows us to take shortcuts when disseminating these results, both to other mathematicians in the field, but also to non-mathematicians. This is especially true of the following result of the author and Hugo Parlier [2].

Theorem (McLeay-Parlier)**.** *Every non-compact topological surface has an ideal triangulation.*

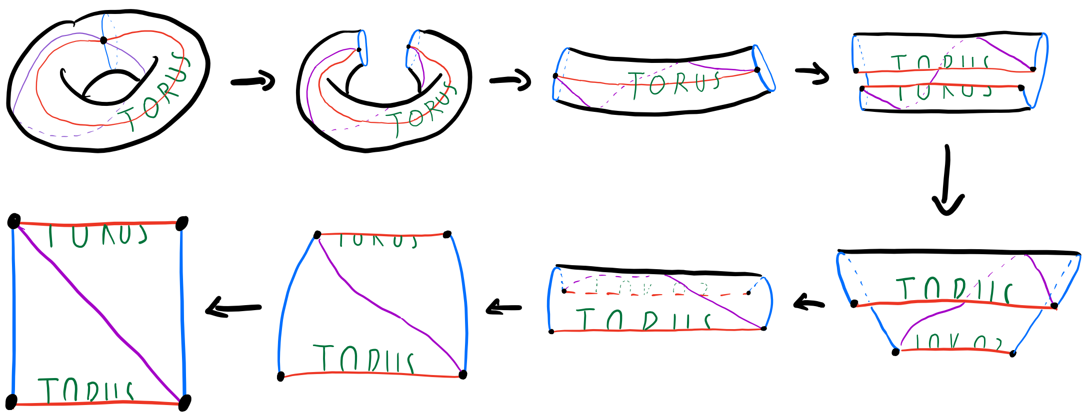
While the formal definitions of the terms in this theorem, and indeed its proof, require a firm understanding of topology, the intuition behind the result can be explained relatively efficiently if we use pictures. Of course, using pictures alone without explanation would not be efficient. This was the inspiration to try to explain the result using a middle ground of *comic strips*.

There are a few other examples of comic strips being used to explain mathematics, for example [3]. A nice additional feature to this medium is that the argument is naturally broken up into its core ideas. This is useful in helping both the reader and the author understand the mathematics. I believe that there is a larger space for this form of outreach literature which can very quickly enable a wide audience to engage with contemporary research.



The objects of study in this project are topological surfaces, and so the drawings in the comic are intended to immediately communicate to the reader that we are in the realm of abstract mathematical shapes. This is not a rigorous introduction to topology however, so the definitions are left to the readers intuition. For example, when the comic speaks of **ends** we use some perspective in the artwork to give a feeling of what an end is.

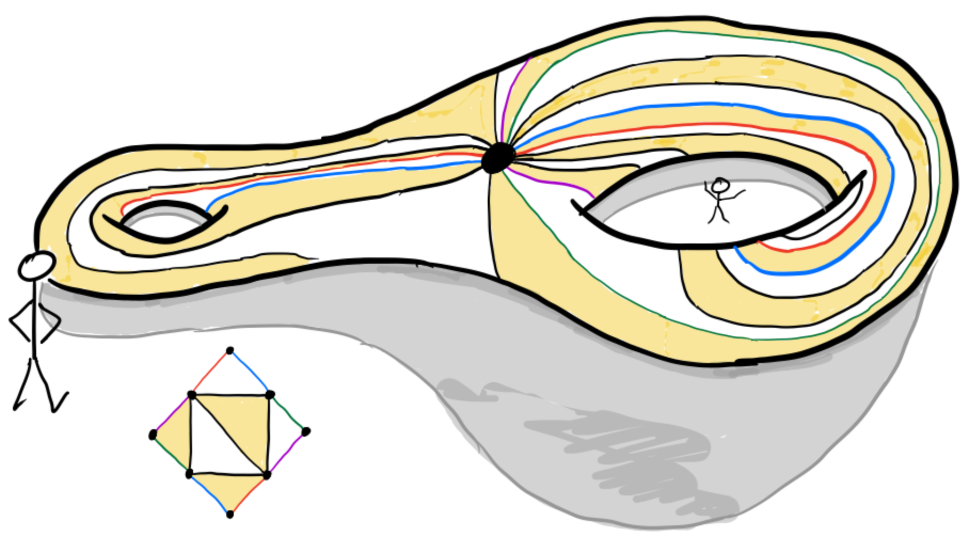
Now that the reader has a vague idea of what surfaces and ends are, the comic moves onto triangulations. In particular, ideal triangulations; these are maximal collections of pairwise disjoint essential arcs (embeddings of the the unit interval not homotopic to the trivial embedding). First, we show the standard unraveling of the torus; equivalently, given two triangles, we can glue them together to make a torus. This is usually shown by triangles attached together in the plane to make a square, then opposite edges of the square are given the same color to indicate the gluing procedure.



The comic then shows how we can obtain higher genus surfaces by gluing more triangles. In particular, we glue one triangle to each edge of the square, so the number of triangles increases by 4, then 8, then 16, etc. The genus, on the other hand, also doubles. So a gluing of

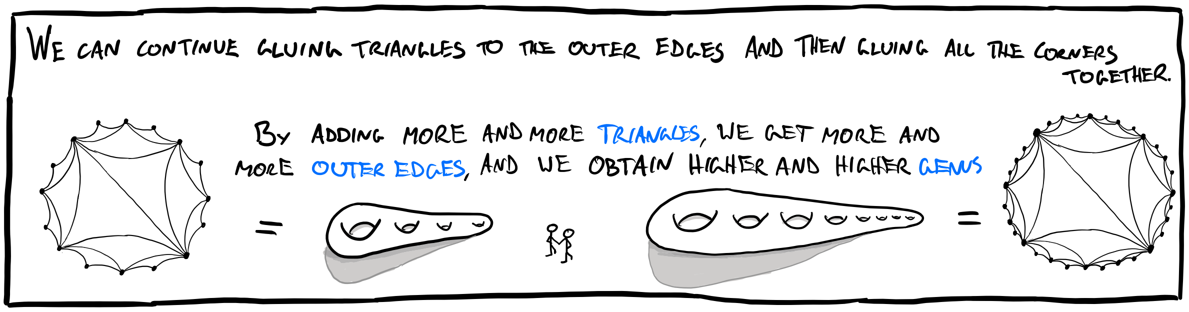
$$2 + 4 + 8 + \cdots + 2^n$$

triangles following this procedure results in a surface of genus 2^{n-1}, along with a triangulation of this surface. This connection between the number of triangles and the genus is not explained in the comic. The comic does show the first four surfaces in this sequence however, so the pattern is there to be observed, even if not explicitly stated.

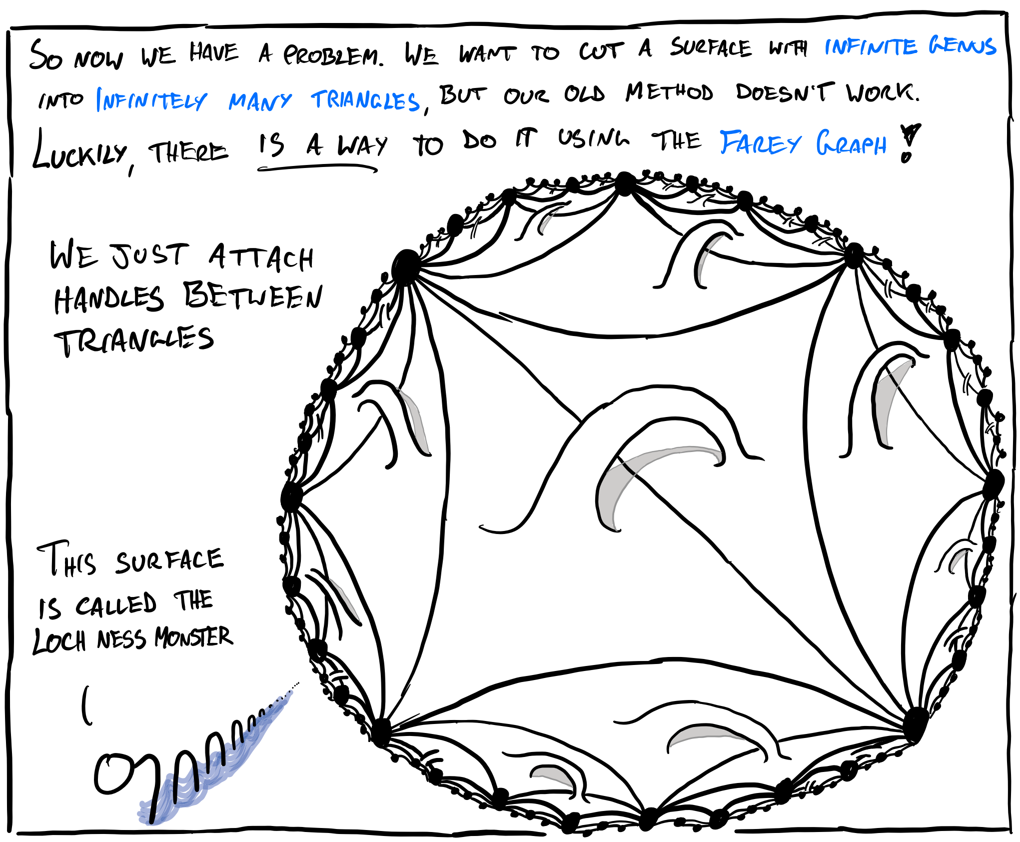


Note Pictures like this are often used by mathematicians working in and around the topology of surfaces. As mentioned above, this can be in the dissemination of their work, but also during an ongoing project. They serve as both a landscape to gather intuition, and as "toy examples" to stress test ideas.

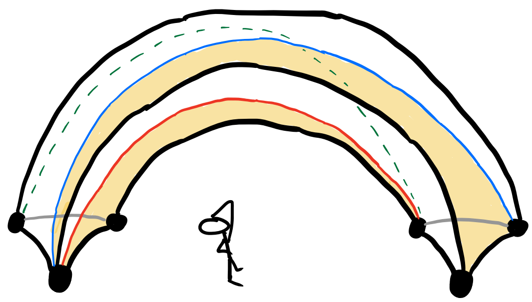
After doing this multiple times, the flat triangulations start to resemble Farey graphs, which are a key tool in the proof of our theorem. Indeed, gluing triangles together in this way is a reasonable way to define a Farey graph; it is the outcome of repeating this procedure infinitely many times.



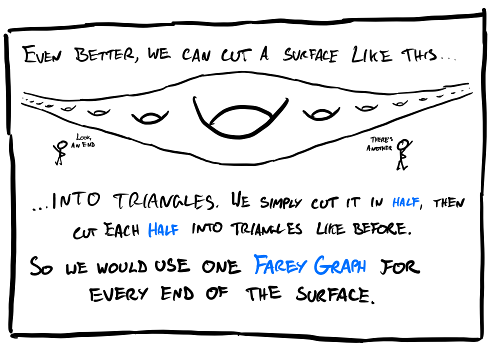
At this point in the comic strip, we make the conceptual leap from finite type surfaces (those whose fundamental groups are finitely generated) to infinite type surfaces (those whose fundamental groups are not finitely generated). In this case, we deal with surfaces of infinite genus. The comic does not define what a fundamental group, or group generation is. Of course, as the reader now understands the notion of a triangulation, we can use these as a bridge to take us into this new realm. We take the now familiar picture of the Farey graph and adorn it with handles between each of the triangles.

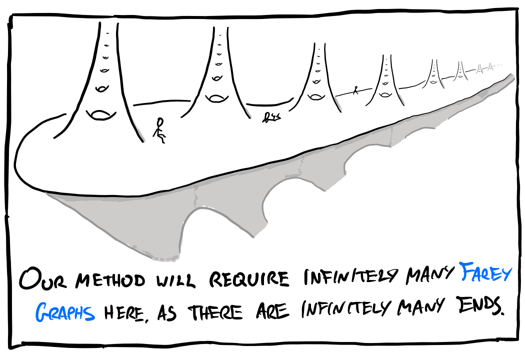


The comic strip ends with a brief description of why this infinitely handled Farey graph is, in fact, a one ended surface (The Loch Ness Monster). It also gives some instruction on how to complete the Farey graph into an ideal triangulation.



We then end with pictures of two more infinite type surfaces which are slightly more complicated. In particular, these surfaces have two and infinitely many ends respectively.





Hopefully by engaging with the piece, the reader has some understanding of what is meant by topology, surfaces, triangulations, ends, etc. Moreover they may well have picked up enough intuition about the subject to convince themselves that these, and indeed all, non-compact surfaces have ideal triangulations.

As mentioned above, proofs and comics share the fact that they tell their stories in discrete steps. Along with outreach, this could be a useful aspect to consider when teaching mathematics. Indeed, an interesting activity for a class learning proof writing could be to translate a proof into a comic strip.

Acknowledgements

The author would lke to thank Hugo Parlier for his enthusiastic response to the first draft of this comic, and for the suggestion to submit to Bridges. He is also grateful to the referee who suggested using this article as a means to elaborate on the intent of the comic.

References

[1] Alan McLeay. Ideal triangulations of surfaces. *https://amcleayblog.files.wordpress.com/2022/01/triangles-comic.pdf*, 2022.

[2] Alan McLeay and Hugo Parlier. Ideally, all infinite type surfaces can be triangulated. *arxiv:2102.09531*, 2021.

[3] Jean-Pierre Petit. *Le Topologicon*. Belin, 1985.