3D Printed Models of Wild Spheres

Lucien Grillet

University of Rennes 1, France ; lucien.grillet@univ-rennes1.fr

Abstract

We discuss two wild embeddings of the sphere in the Euclidean space: the Alexander horned sphere and the Fox-Artin sphere. After explaining the mathematical properties carried by these two embeddings, we present 3D printed models of these objects. The models have been designed to be decorative table-top sculptures while being simple enough to serve as supports for mathematical discussions.

1 The Alexander horned sphere

In topology, the famous Jordan-Brouwer separation theorem states that if one draws a closed line on the plane which does not intersect itself, this line separates the plane into two disjoint regions.

Theorem (Jordan-Brouwer separation theorem)**.** *Let C be a continuous simple closed curve in $\mathbb{R}^2$. Then $\mathbb{R}^2 \setminus C$ has exactly two connected components.*

This theorem can seem a bit weak: we know that the line separates the plane into two regions but we do not know what the topology of these two regions are. In fact, the Jordan-Schoenflies theorem tells us that such a line can be continuously transformed to a standard circle.

Theorem (Jordan-Schoenflies theorem)**.** *Let C be a continuous simple closed curve in $\mathbb{R}^2$. There is an homeomorphism of $\mathbb{R}^2$ into itself that sends C on the standard circle $\left\{x \in \mathbb{R}^2 \mid \|x\|_2 = 1\right\}$.*

This theorem is stronger than the Jordan-Brouwer separation theorem because it completely describes the situation. In particular it shows that the region bounded by the line must be homeomorphic to an open disk.

Going one dimension higher, one can wonder if the situation is similar for embeddings of the sphere S^2 in the Euclidean space $\mathbb{R}^3$. A first positive result is the Alexander duality, which can be thought of as a generalization of the Jordan-Brouwer separation theorem, it can be used to show that any continuous injective map from the sphere S^{n-1} into the sphere $\mathbb{R}^n$ cuts the latter into two pieces.

However, the Jordan-Schoenflies theorem does not generalize to higher dimensions. *The Alexander horned sphere* (Represented in Figure 1) is a subspace of $\mathbb{R}^3$ homeomorphic to a sphere but whose exterior (together with the point at infinity) is not homeomorphic to an open ball. Indeed, its exterior is not even simply connected (the curve γ on Figure 1 cannot be shrunk to a point).

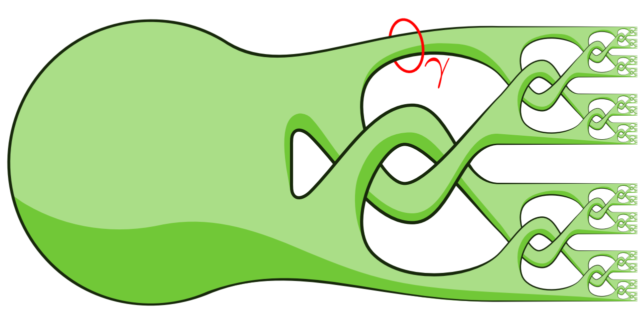

Figure 1: *The Alexander horned sphere*

This sphere was first described by J. W. Alexander in [1] and is an example of what we call a *wild embedding*: it is a continuous embedding such that there is no homeomorphism of $\mathbb{R}^3$ into itself that sends it to a smooth embedding. In fact, in this representation, the sphere is smooth everywhere, except on a Cantor set around which the sphere makes an infinite number of knots.

To construct this object, start with a sphere and grow two horns out of it. Divide each of these horns into two other horns and interlace one horn of the first division with one horn of second division. Repeat this process with each new pair of horns over and over infinitely many times with smaller and smaller horns until they accumulate on a Cantor set.

The resulting object is homeomorphic to a sphere, but if one defines a loop around one of the first horns, the infinite number of interlacing makes it impossible to shrink this loop to a point.

To make this object easier to understand, I decided to recreate it in the real world. Thanks to a 3D modeling software, I designed a model of the Alexander horned sphere. I chose a smooth embedding (except on the singular Cantor set) for the visual aspect and I chose to represent four levels of the pattern, with a scaling factor of 43% between each level, to obtain a model with as many levels as possible while being printable with the precision and the size of the 3D printer. The result is the 20 centimeters object shown in Figure 2. The main difficulty in the printing process was the fact that the printer needed to print a support together with the sphere, because the PLA filament always needs to be deposited on a solid structure. I chose to print this support with a soluble PVA filament to obtain a smooth result without any trace of this support.

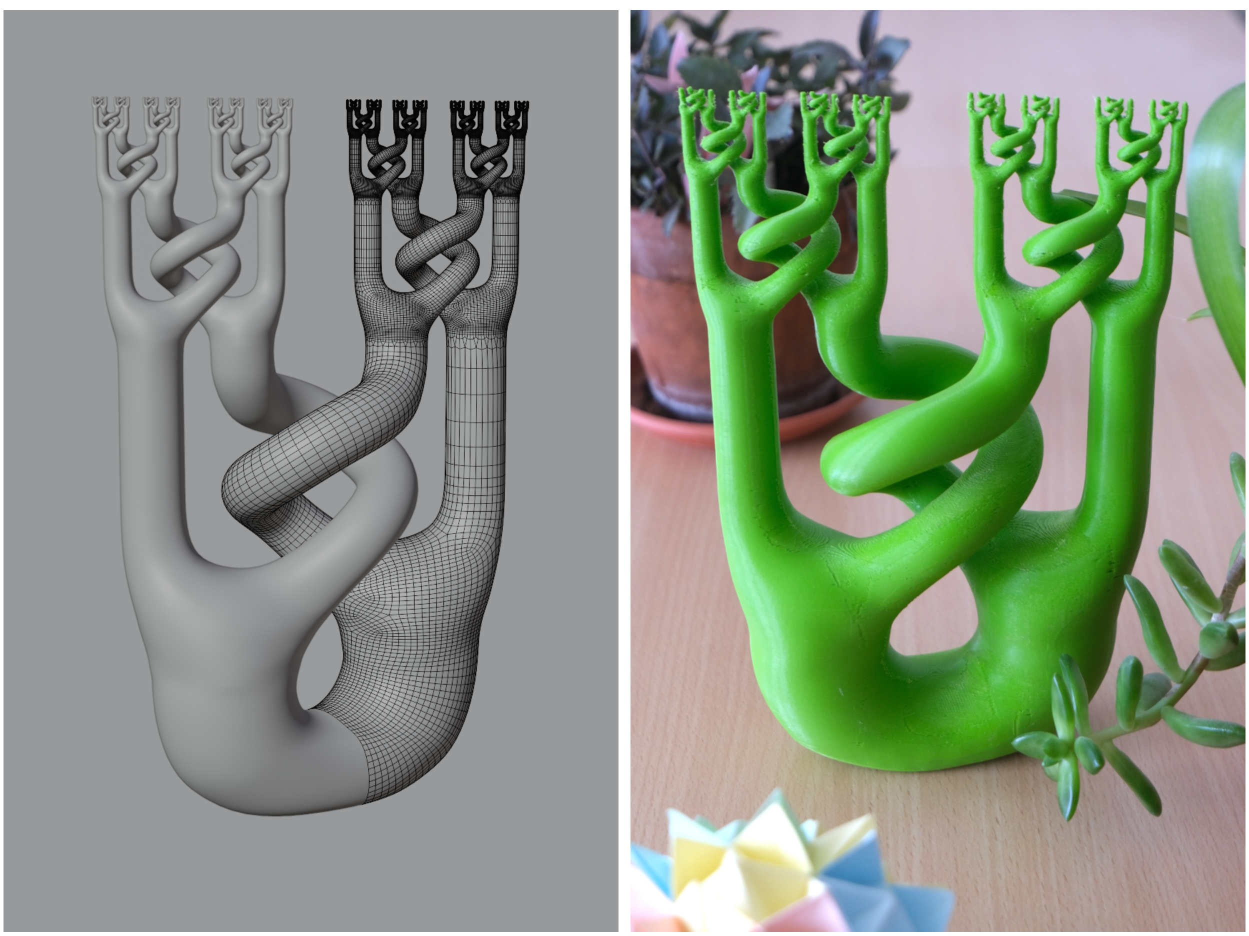

Figure 2: *3D model and 3D printed sculpture of the Alexander horned sphere*

2 The Fox-Artin sphere

The Alexander horned sphere is well-known, but a wide diversity of other wild spheres exist without having the same reputation. A reason for this is the lack of visualization of these spheres. Among all of them, there is one that I find particularly interesting: *the Fox-Artin sphere* (represented on Figure 3). I first encountered this sphere when confronted to the following question during my research work.

Question. *Let K be a compact subset of* $\mathbb{R}^3$*, and suppose that K verifies the two following properties.*

- *The interior* $\mathring{K}$ *of K is homeomorphic to an open 3-ball*
- *The boundary* ∂K *of K is homeomorphic to a 2-sphere*

Is K homeomorphic to a closed ball ?

The Fox-Artin sphere offers a negative answer to this question: it is a subset of S^3 homeomorphic to a 2-sphere and it cuts S^3 into two components homeomorphic to open 3-balls. However, the union of the exterior component (according to Figure 3) with this 2-sphere does not yield a closed ball.

Returning to the situation in dimension 2, the Jordan-Schoenflies theorem told us two things:

(A) The interior of a simple closed curve is homeomorphic to an open disk.

(B) The union of a simple closed curve with its interior is homeomorphic to an closed disk.

In other words, it says that the homeomorphism from the open disk to the interior of C can be chosen so that it extends homeomorphically to C.

The Alexander horned sphere contradicts the point A (and thus the point B) while the Fox-Artin sphere only contradicts the point B. The answer to our question is thus negative: it is strangely possible to "glue" a 2-sphere on the boundary of an open 3-ball without obtaining a closed 3-ball.

However, as in the Alexander horned sphere, the interior component together with the sphere yield a subset homeomorphic to a closed ball.

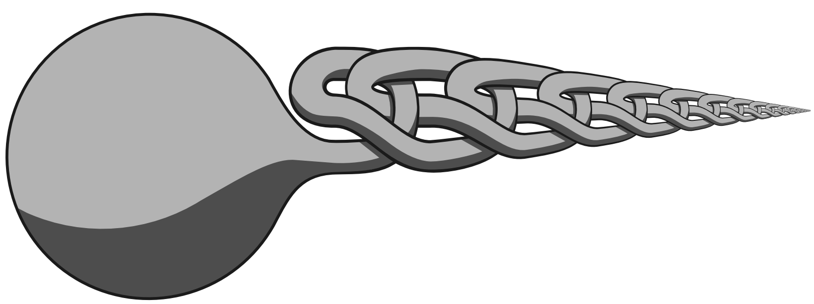

Figure 3: *The Fox-Artin sphere*

This sphere is defined as a thickening of a wild arc, with a thickness converging to zero as we approach the endpoint. The arc used here is an infinite concatenation of loops where each loop is knotted with the previous one. This arc has been defined by R. H. Fox and E. Artin in [2]. In fact, the term "Fox-Artin sphere" can refer to slightly different objects, as several wild arcs have been defined in the aforementioned article. Here, we only consider the simplest of these arcs, which is infinitely knotted only on one side, but we could construct other wild spheres with other properties by working with other arcs.

Compared to the Alexander horned sphere, this sphere is usually easier to comprehend for people because it has a single singular point. It is clearer that it is homeomorphic to a sphere, but its properties are more difficult to understand and to prove.

I made a 15 centimeters sculpture of this Fox-Artin sphere with the same method used for the Alexander horned sphere. The model is smooth everywhere except at its singular endpoint. To reduce the height of the sculpture and to make a base for the sculpture to stand on, I decided to flatten the original sphere from which the horn is grown. With the precision of the 3D printer, I could print seven levels of the pattern with a scaling factor of 67% between each level. Figure 4 represents the 3D model and the final result.

The files used in this article can be freely accessed on http://lugri.net/maths/wildspheres.php.

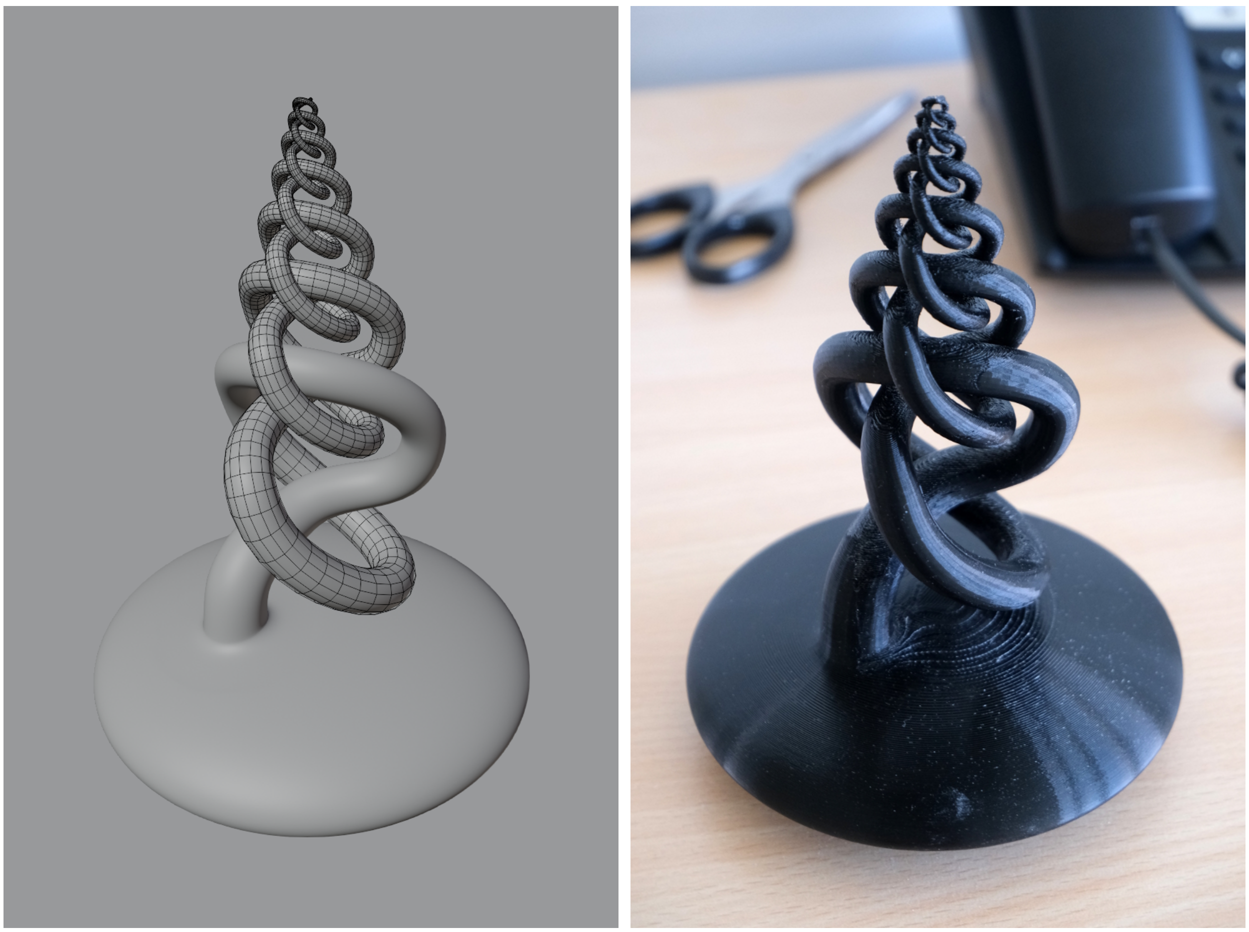

Figure 4: *3D model and 3D printed sculpture of the Fox-Artin sphere*

Acknowledgements

Thanks to Rémi Coulon who helped me with the technical difficulties involved in the process of printing such knotted objects.

References

[1] J. W. Alexander. "An Example of a Simply Connected Surface Bounding a Region which is not Simply Connected." *Proceedings of the National Academy of Sciences of the United States of America*, 1924.

[2] R. H. Fox and Emil Artin. "Some wild cells and spheres in three-dimensional space." *Annals of Mathematics, Second Series*, 1948.