Prime Factorization Fractal Tilings

Santo Leonardo

Milan, Italy; santo.leonardo496@gmail.com

Abstract

This paper introduces an approach for creating tilings made by the iterative fragmentation of polygons according to the prime factors of any integer greater than 1 \((n)\), thus attempting to link the properties of \(n\) to a tessellation made of \(n\) tiles. Choosing \(n\) as the power of a base integer, the method produces non-hyperbolic bounded fractal tilings, rich in visual complexity. In addition, the paper presents a colouring method, which iteratively applies the prime factorization fractal approach to the luminosity dimension, thus synchronising colour and geometry along the iterative tiling process.

Prime Factorization Tilings

As I am fascinated by Number Theory, in order to visualize number properties, I explored the idea of fragmenting regular polygons into tessellations according to the prime factors of an integer \([2]\). The proposed tiling approach starts from selecting an integer \((n \geq 2)\), and decomposing it into its prime factors sorted for example in ascending order \((n=p_1 \cdot p_2 \ldots p_k)\). Then it selects an initial regular polygon, whose number of sides is function of \(p_1\). Afterwards the initial shape is fragmented into \(p_1\) tiles, similarly to “splitting a pie”. Each resulting tile is then furtherly fragmented into \(p_2\) tiles, and the process is iterated until the prime factors of \(n\) are exhausted: the substitution rule at level \(i\) is determined by \(p_i\). The fundamental theorem of arithmetic ensures that each \(n\) produces a unique tiling structure, made of \(n\) tiles (Figure 1).

In this paper I extend my idea by investigating the case where \(n\) is the power of an integer base \((n=b^l)\) and produces a fractal as the exponent \((l)\) increases. In order to reach this objective, I adopt a new approach for the iterative tiles splitting, suited to fractals; and I introduce a colouring method, which consists of a one-dimension prime factorization fractal synchronised with the geometry.

![Figure 1: Prime factorization of (a) \(n=4025=5\cdot5\cdot7\cdot23\) and (b) \(n=4068=2\cdot2\cdot3\cdot3\cdot113\), from[2].](image)

The Geometry of Prime Factorization Tilings

In order to split a polygon (tile \(T_j\), where \(j\) is the tile index) with \(e\) edges into \(p_i\) sub-tiles, I apply four steps:

A) Select a central point within the polygon (central vertex \(V_c\)), e.g. the barycentre

B) Define a reference point on the tile border (synch point \(V_0\)). In this paper I choose the vertex \(V_c\) of the parent tile \(T_p\) (tile from which \(T_j\) was originated), i.e. \(V_0(T_j)=V_c(T_p)\)
C) Create the sub-tiles by splitting the border in $p_i$ equal parts (split metric), starting from $V_0$ and connecting each two selected consecutive vertices to $V_i$. In some variations the starting point can be taken at a defined distance from $V_0$ (shift $S_0$) producing alternative interesting configurations.

D) When necessary, normalize the resulting tiles order by appropriately adding vertices on the edges (tile normalization).

The above steps A-D are jointly defined in order to reach two objectives that contribute to the final symmetry of the image: (Ob1) adjacent tiles are split so that their sub-tiles are aligned on vertices’ positions along the common border, (Ob2) the produced sub-tiles have a coherent vertex numeration, therefore enabling a coherent splitting and colouring in the following iteration phases.

The driving choice is the definition of the split metric (C). A natural method would be to divide the tile perimeter in equal $p_i$ parts using Euclidean metric ($E\langle P_i, P_j \rangle$ in the following), however this approach does not align the sub-tiles vertices during the iteration process (Ob1), see Figure 2 (a) for $n=455$. In [2] I introduced a specific metric, defining the distance of two points $P_1, P_2 \ (GE\langle P_1, P_2 \rangle)$ as:

$$GE(P_1, P_2) = E(P_1, P_2)/E(V_i, V_i), \ \text{if } P_1 \ \text{and } P_2 \ \text{are on the same edge delimited by } V_i, V_i$$

$$GE(P_1, P_2) = GE(P_1, V_i) + k + GE(V_i, P_2), \ \text{if } P_1 \ \text{and } P_2 \ \text{are separated by vertices } V_i, V_{i+1}...V_{i+k}.$$

The proposed metric $GE$ combines Graph and Euclidean distances, and requires an ordering of the vertices, which is carried out in step (B). This approach has to be used together with the normalization of all sub-tiles to order 4, which is done for triangles by adding a vertex $V$ such that $GE(V_0, V)=1.5$, that is by adding a vertex at the midpoint of the external edge vs $V_0$. The final result is shown in Figure 2 (b) for $n=435$. Interesting patterns are obtained also by using a shift $S_0=e/2$, which preserves the overall symmetry.

Figure 2: Tiling of $n=455=5\cdot7\cdot13$ by approach: (a) Euclidean, (b) Graph-Euclidean, (c) Modular-3, (d) Modular-4.

A further approach is to split a tile with $e$ edges in $p_i$ sub-tiles by assigning to each edge $p_i/e$ (quotient) sub-tiles, and distributing the remaining $p_i$ modulo $e$ (remainder) sub-tiles appropriately in order to keep the symmetry. For example, for $e=3$ and $p_i=5$ by assigning the remainder 2 tiles, one to the edge $V_{e}V_1$ and one to the edge $V_2V_0$, and for $e=3$ and $p_i=7$ by assigning the remainder 1 tile to the edge $V_7V_2$.

I investigated two alternatives: 1) “Modular-3” in which all created tiles are triangles, achieved by ensuring that at each step sub-tiling always includes the vertices of the parent tile (Figure 2-c); 2) “Modular-4” in which any created tiles has order 4, achieved by normalizing triangles into quadrilaterals, e.g. by adding an additional vertex at point $P$: $GE(V_0, P)=1.5$ (Figure 2-d). Modular-4 approach is employed by adopting a shift $S_0=0.5$ in (GE metric). In both approaches a specific splitting choice is required for $p_i=2$.

Prime Factorization Fractals Tilings

Choosing $n=b^l$, where $b=p_1 p_2... p_k$, and sorting the factors of $n$ as $(p_1...p_k, p_1...p_k, ...)$ “power-ordering”, the tiling process produces a fractal as the integer $l$ (levels) increases; indeed, after that the initial polygon is split according to the factors of $b$ (first level, $l=1$) each resulting tile will be further split according to the factors of $b^{l-1}$, producing a self-similar non-hyperbolic bounded tiling [1] (Figure 3).
The Modular approach, introduced in this paper, produces rich patterns for many \( b \) fractals, see Figure 4 for \( n=15^3=(3\cdot5)^3 \) created with modular-4 and Graph-Euclidean approaches all other things being equal (using Binomial colour Intensity distribution, see further explanations).

The subcase \( n=3^l \) in Modular-3 approach, with a shift \( S_0=0.5 \) and eliminating the central vertex, e.g. directly connecting the new vertices during the splitting process, is equivalent to the Sierpinski triangle.

From “Colouring the Fractal” to “Fractalizing the Colour”

In the previous artwork [2], based on sorting the factors \( p \), in ascending-order, I coloured the tiling in the Hue, Saturation and Intensity colour space \((H, S, I)\) by defining \( H \) and \( I \) as functions of the tile index modulo the product of a subset of the prime factors of \( n \). I selected larger factors for \( I \) and smaller ones for \( H \): this approach puts in evidence the details through the luminosity, while providing an overview of the tessellation distribution through the Hue (Figure 1). \( S \) is defined as a linear function of \( I \).

In the case of fractals, where a power-ordering is adopted, a different approach is necessary in order to maintain the structure self-similar while increasing \( l \). I propose to produce a 1-dimension fractal of \( I \), synchronised with the geometric splitting process. I assign iteratively the intensity to the tiles: given \( n=b^l \), once defined the intensity \( I_a \) of a tile at the splitting iteration level \( k \) \((k=1...l)\), the intensities of the corresponding sub-tiles \( I_x \) \((x=1...b)\) at level \( k+1 \) are defined such as \( \text{average}(I_x) = I_a \). This approach preserves the Intensity self-similarity through the iteration levels (Figure 5). I explored different distributions for the \( I_x \) values: e.g., linear, circular and binomial, in the last case for odd \( b \) assigning the average \( I_a \) to one tile. The distribution range used at each step has to be approximately normalised in order not to exceed the overall definition range of \( I [0, 1] \), e.g. using a multiple of the distribution standard
deviation calculated over \( l \) levels. Finally, different distributions can be mixed by a weighted sum for interesting artistic effects.

Figure 5: Tiling of \( n = 5^3, 5^4, 5^5 \). Circular Intensity distribution, Modular-4 approach.

For the proposed fractals I maintain the same definition of \( S \) and \( H \) used for the ascending ordering [2], as it has a consistent behaviour for fractals (Figure 6).

Figure 6: Tiling of \( n = (a) \ 3^8, (b) \ 10^4 \). Binomial Intensity distribution, Modular-4 approach.

Conclusions

In this paper I attempted to build a bridge between the “hidden” beauty of Number Theory and the “evident” elegance of Tilings through fractal tessellations based on the structure of the integers’ prime factorization, which acts both on geometry and colours dimensions. Several combinations of the described sub approaches can be used, resulting in different families of infinite tilings; I prefer some parameter sets depending on the used integer base. There are additional possibilities to be investigated, e.g.: fractalize also the Hue dimension, explore different ways for tiles normalization e.g. to order 4, represent integers’ properties through tiles colouring e.g. totient function.

References
