# Deltoidal kaleidoscopes

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## Abstract

We present deltoidal kaleidoscopes that allow one to visualize not only the Platonic and the Archimedian solids, but also their duals. They were premiered in the exhibition "Imaginary, a Mathematical Symphony" hosted by the Universitat Politècnica de Catalunya (Barcelona, September-December 2021).

## Polyhedra and kaleidoscopes

Regular polyhedra have many planes of symmetry. By placing mirrors on those planes, we can generate kaleidoscopes which, from a fraction of a polyhedron, allow us to visualize the whole object.

For example, if we use mirrors to build the "baseless inverted pyramid" that connect the center of a regular polyhedron with one of its faces, and then place a hollow face on top, we can visualize the whole polyhedron. These are called regular kaleidoscopes and they "multiply" by 4, 6, 8, 12 or 20 depending on which Platonic solid we use.

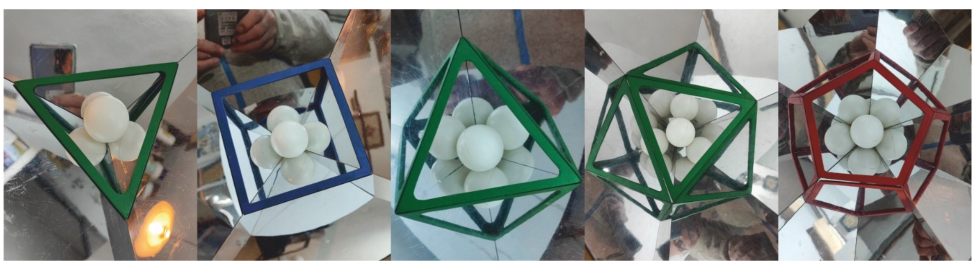


**Figure 1:** *The regular kaleidoscopes that result from the five Platonic solids*

## Minimal kaleidoscopes

There are many types of kaleidoscopes that are able to reproduce a given solid, so it makes sense to look for the smallest ones, and then combine these minimal kaleidoscopes to obtain the other families.

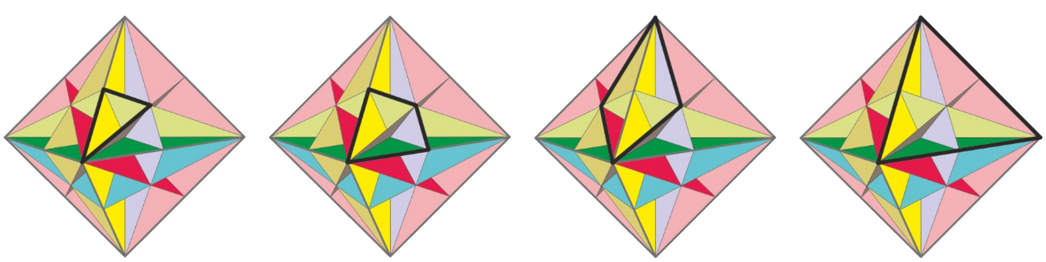


**Figure 2:** *Minimal, deltoidal, rhombic and regular octahedral kaleidoscopes.*

We have briefly described the regular kaleidoscopes and we use the rhombic kaleidoscopes in the permanent exhibition of the Catalan Museum of Mathematics (MMACA). In this paper we want to introduce the deltoidal kaleidoscopes, but in order to do so we need to take a closer look at the minimal kaleidoscopes and study their properties.

***Minimal octahedral kaleidoscope***

Let's start with the octahedron. It has 9 planes of symmetry. As shown in Figure 3 they divide each face into six equal triangles, and the octahedron into 48 equal pieces. Each piece is an inverted pyramid that has its apex at the center of the octahedron and its base on each of the aforementioned triangles.

Therefore, the smallest part of the octahedron that allows the whole octahedron to be reconstructed by symmetries is 1/48th of the total and we could use it to create a minimal octahedral kaleidoscope with dihedral angles of 45º, 60º, and 90º. The edges of dihedral angles are called S8, S6 and S4, respectively.

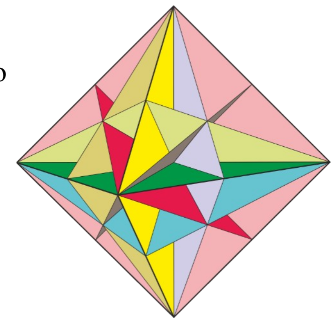


**Figure 3:** *Octahedron*

The octahedron and the cube have the same planes of symmetry, which means that the cube could also be decomposed into 48 equal pyramids with the same dihedral angles but a different base. Figure 4 shows how the S4, S6 and S8 edges of the cubic and the octahedral pyramids are parallel and therefore give rise to the same kaleidoscope.

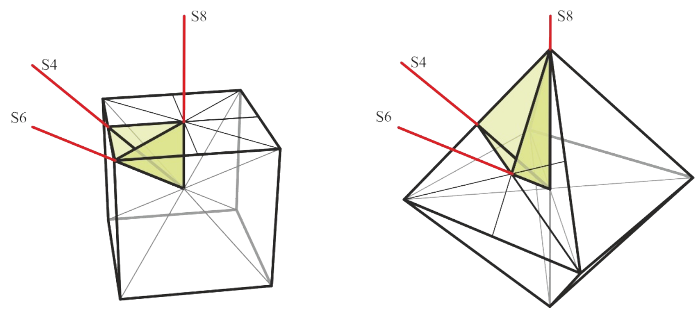



**Figure 4:** *Minimal octahedral kaleidoscope*

In this minimal kaleidoscope, the cube will be obtained with a "half half-edge" perpendicular to S4 on the opposite side of S8 whereas the octahedron will require a "half half-edge" perpendicular to S4 but on the opposite side of S6. These two edges meet at a point of S4 which shows the duality between these solids. Since in both cases we use 1/4 of an edge, and the minimal octahedral kaleidoscope multiplies by 48, the 12 edges of the cube and the octahedron will be generated.

***Minimal icosahedral kaleidoscope***

The dodecahedron and the icosahedron also share their 15 planes of symmetry, and minimal kaleidoscopes can be obtained from 1/120th of these solids. In the first case, each face gets divided into 10 equal triangles, whereas in the second case each face can be divided into 6 equal triangles. Both kinds of pyramids have the same dihedral angles of 36º, 60º, and 90º and their corresponding edges are called S10, S6, and S4. (Figure 5)

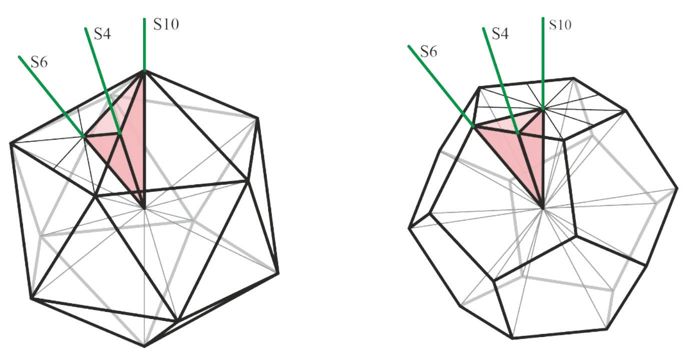



**Figure 5:** *Minimal icosahedral kaleidoscope*

In this minimal kaleidoscope, the dodecahedron will be obtained with a "half half-edge" perpendicular to S4 on the opposite side of S10 whereas the icosahedron will require a "half half-edge" perpendicular to S4 on the opposite side of S6. These two edges meet at a point of S4 which shows the duality between these solids. Since in both cases we use 1/4 of an edge, and the minimal icosahedral kaleidoscope multiplies by 120, the 30 edges of the dodecahedron and the icosahedron will be generated.

## Deltoidal kaleidoscopes

We can focus, at last, on the main subject of this paper, the deltoidal kaleidoscopes that result from joining a pair of minimal kaleidoscopes by their "hypotenuse" (the face opposite to the right dihedral angle).

***Deltoidal icositetrahedral kaleidoscope***

From the combination of two minimal octahedral kaleidoscopes (1/48), we obtain a deltoidal icositetrahedral kaleidoscope (1/24) with three dihedral angles of 90º and a fourth one of 120º. The two S4 edges keep their names while the union of two S8 axes is now called D4 and the union of two S6 edges is now called D3. The base that is perpendicular to the intersection of the diagonal planes of this kaleidoscope is a deltoid. More precisely, it is the face of the Catalan solid deltoidal icositetrahedron (dual of the small rhombicuboctahedron). (Figure 6)

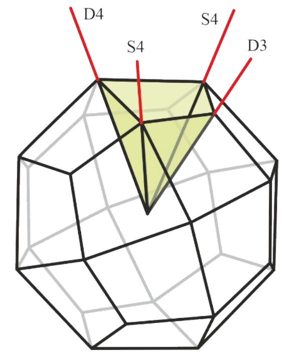



**Figure 6**: *Deltoidal icositetrahedral kaleidoscope*

The cube is generated with a piece formed by two "half half-edges" at 90º, with the vertex on the D3 edges and in a plane perpendicular to D4. The careful construction of the pieces, with the right slope, makes them fit naturally into place. Since this piece contains "1/3 of the vertex" and two quarters of the "edge", and the deltoidal icositetrahedral kaleidoscope multiplies by 24, the 8 vertices and 12 edges are generated.

We generate the octahedron analogously with two "half half-edges" at 60º with the vertex on the opposite D4 edges and in a plane perpendicular to D3. (Figure 7)

The third piece in the Figure 8 generates the intersection of the cube and the octahedron: a cuboctahedron.

**Figure 7:** *Cube and octahedron*

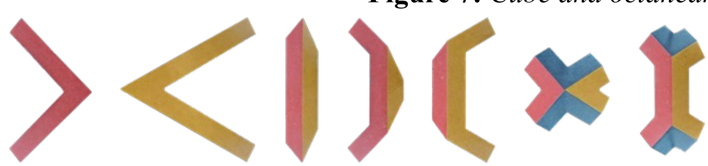
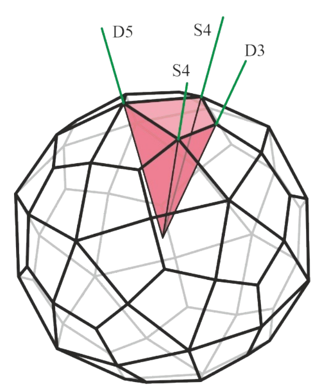


**Figure 8:** *The pieces that generate the Platonic and Archimedean solids of the octahedral group in the deltoidal icositetrahedral kaleidoscope must be symmetrical and have this appearance.*

***Deltoidal hexacontahedral kaleidoscope***

From the combination of two minimal icosahedral kaleidoscopes (1/120), we obtain a deltoidal hexacontahedral kaleidoscope (1/60) with dihedral angles of 90º, 120º, 90º and 72º. The dihedral axes are named, respectively, S4, D3, S4 and D5. The base that is perpendicular to the intersection of the diagonal planes of this kaleidoscope is the face of a deltoidal hexacontahedron (dual of the rhombicosidodecahedron).



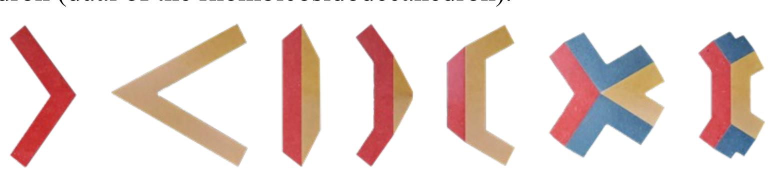



**Figure 9:** *Deltoidal hexacontahedral kaleidoscope and its pieces*

In this deltoidal kaleidoscope, the dodecahedron is generated by a piece formed by two “half half-edges” at 108º, with the vertex on the D3 axis and in a plane perpendicular to D5. Since this piece contains “1/3 of the vertex” and 1/2 of the “edge”, and the deltoidal hexacontahedral kaleidoscope multiplies by 60, the 20 vertices and the 30 edges are generated.

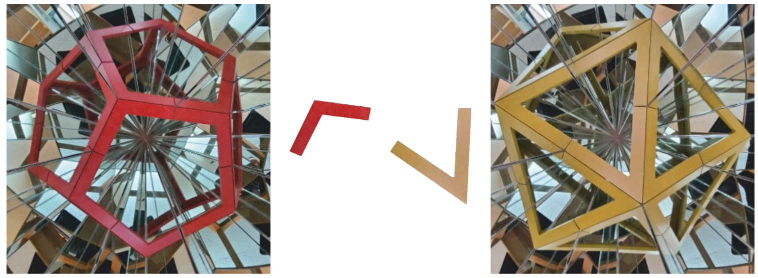


**Figure 10:** *Dodecahedron and icosahedron*

We generate the icosahedron analogously with two “half half-edges” at 60º, with the vertex at the D5 axis and in a plane perpendicular to D3. The third piece in the picture above allows us to see the icosidodecahedron as the intersection of the icosahedron and the dodecahedron. (Figure 11)

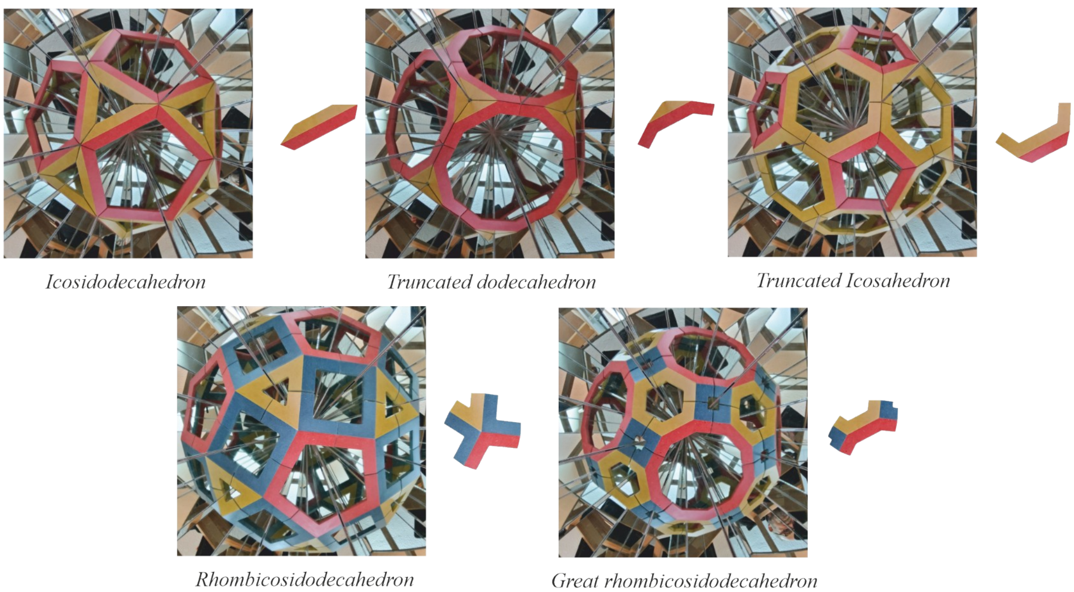



**Figure 11:** *The five Archimedian polyhedra in the deltoidal hexacontahedral kaleidoscope*

More information about kaleidoscopes and related topics in this paper can be found in [1-5].

## References

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