# Thales' Theorem, Pythagorean Triples, and Geometric Art

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## Abstract

The authors present and prove the correctness of a geometric construction, stemming from Thales' Theorem, that produces rectangles with length-to-width ratio $(n + 1) : n$, for any positive integer $n$. These superparticular ratios, which include the most basic musical ratios, are important to those who study the relationships between art and math. The construction inherently involves a well-known family of Pythagorean triples, potentially in a novel way. The authors, one of whom is an artist, also present some examples of artwork inspired by Thales' Theorem.

## Introduction

The authors of this article, a mathematician and an artist, have been collaborating for many years [6, 11, 9, 10], but the mathematics of this present collaboration is the most significant to date. The artist, Reynolds, who taught geometric systems for over 25 years at the Academy of Art University in San Francisco, is represented by Pierogi Gallery in New York. His second show there, in June 2015, inspired a review in *The New Yorker* magazine acclaiming "the exquisite drawings of this San Francisco-based artist, who, armed with straightedge and compass, transmutes the mysteries of geometry into dense meshes of colored lines, alive with spiritual intensity" [3]. Examples of his artwork will be included below; examples of his scholarship include [7, 8].

Reynolds often discovers interesting mathematical relationships within his geometric constructions. When he wants to verify them, he calls on Wassell, a retired math professor whose research is primarily on the math/art interface, often pertaining to architecture, e.g., [4, 14, 12]. When Reynolds contacted Wassell about one of his latest series of geometric constructions, Wassell discovered that the underlying mathematics involves arguably the most well-known family of Pythagorean triples. The construction is so simple that one would think it must have been observed before, but searches for it in the literature have come up empty.

## Thales' Theorem

The inspiration for all of the art and math in this article is Thales' Theorem, which states that an inscribed angle subtended by the diameter of a circle must be a right angle. Reynolds observed that it is possible to construct "all the rectangles of the world" (as he puts it, meaning all possible length-to-width ratios) simply by choosing the appropriate point on the circumference at which the inscribed angle occurs (point $P$ in Figure 1) and then completing the rectangle using symmetry.

Recall that two rectangles are similar if they have the same shape, i.e., if they have the same length-to-width ratio, $\ell : w$, where we choose $\ell \geq w$. For example, all squares are similar (since $\ell : w = 1$, regardless of size), and thus one can speak of "the square" to mean a generic square of any given size. Other rectangles important in the math/art/music world, ordered by $\ell : w$ ratio, would include the major fourth ($4 : 3 \approx 1.333$), the root-2 rectangle ($\sqrt{2} : 1 \approx 1.414$), the major fifth ($3 : 2 = 1.5$), the golden rectangle ($(1 + \sqrt{5}) : 2 \approx 1.618$), and the double square or octave ($2 : 1 = 2$).

Naturally, these were the kinds of rectangles that Reynolds first tried to construct. That is, he sought a method, using straightedge and compass, to locate $P$ on the circumference of the circle to produce the desired

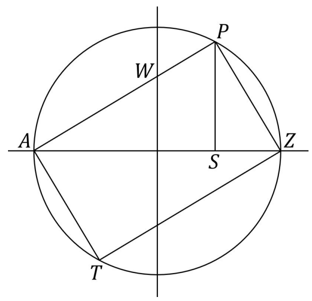


**Figure 1:** *Since the inscribed angle $APZ$ is subtended by the diameter $AZ$, it is a right angle, and then by symmetry $APZT$ is a rectangle.*

rectangle. A square is simple, of course. If we take the circle in Figure 1 to be the unit circle, as we will from now on, so that $A = (-1, 0)$ and $Z = (1, 0)$, then choosing $P = (0, 1)$ (and thus $T = (0, -1)$) clearly produces a square. We then see that we can produce "all the rectangles of the world" by letting $P$ run through all points along the circumference in the first quadrant, including $(0, 1)$ but not $(1, 0)$.

Surprisingly, while Reynolds used or found constructions for some key rectangles involving irrational length-to-width ratios, the musical rectangles (such as $2 : 1$, $3 : 2$, and $4 : 3$) did not reveal any geometric solutions to him (not at first, at least). He asked Wassell to try to find a solution, who opted to think of the problem in an algebraic way and quickly found a very simple rule. Referring to Figure 1, if we want $AP/PZ$ to be equal to any chosen $x \geq 1$, we simply locate point $S$ so that $AS/SZ = x^2$. The problem for Reynolds was that this simple algebraic procedure did not seem to correspond to a simple geometric construction. He continued his own search, though, and eventually found just about as elegant a construction as one can hope for, involving an important family of Pythagorean triples.

## Pythagorean Triples

The famous Pythagorean Theorem perhaps just as famously has integer solutions, such as $(3, 4, 5)$, which are now often referred to as Pythagorean triples. In the *Elements* Euclid showed that one could produce an infinite number of Pythagorean triples using simple formulas (Book X, Lemma 1 to Proposition 29 [2, v. 3, 63]). For a modern formulation, take any positive integers $m > n$ and define $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. As a special case, taking $m = n + 1$ produces the following family of Pythagorean triples:

$$\begin{aligned} a_n &= (n+1)^2 - n^2 = 2n + 1 \\ b_n &= 2(n+1)n = 2n^2 + 2n \\ c_n &= (n+1)^2 + n^2 = 2n^2 + 2n + 1. \end{aligned} \tag{1}$$

We can readily observe that $c_n = b_n + 1$, as is seen in the specific members of the family: $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$, etc. This particular sequence of integer solutions to the Pythagorean Theorem was known by his time [2, v. 1, 356ff]. When Wassell checked the mathematics underlying Reynolds' eventual solution to his quandary, what appeared were these same Pythagorean triples $(a_n, b_n, c_n)$ of Eq. (1), inherently in the points $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ on the unit circle. It is certainly well known (see, e.g., [1]) that the rational points on the unit circle can be parameterized by intersecting lines through $A = (-1, 0)$ of rational slope with the unit circle (see Figure 1). In particular, a line through $(-1, 0)$ having rational slope $t = m/n$ also runs through the point $W = (0, t)$ and corresponds to the rational point $P = \left(\frac{m^2-n^2}{m^2+n^2}, \frac{2mn}{m^2+n^2}\right)$ on the unit circle. However, Reynolds' simple construction for the subset of rational points $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ seems to be missing from the literature, as does his concomitant construction for the "superparticular" ratios, so called for reasons we now review.

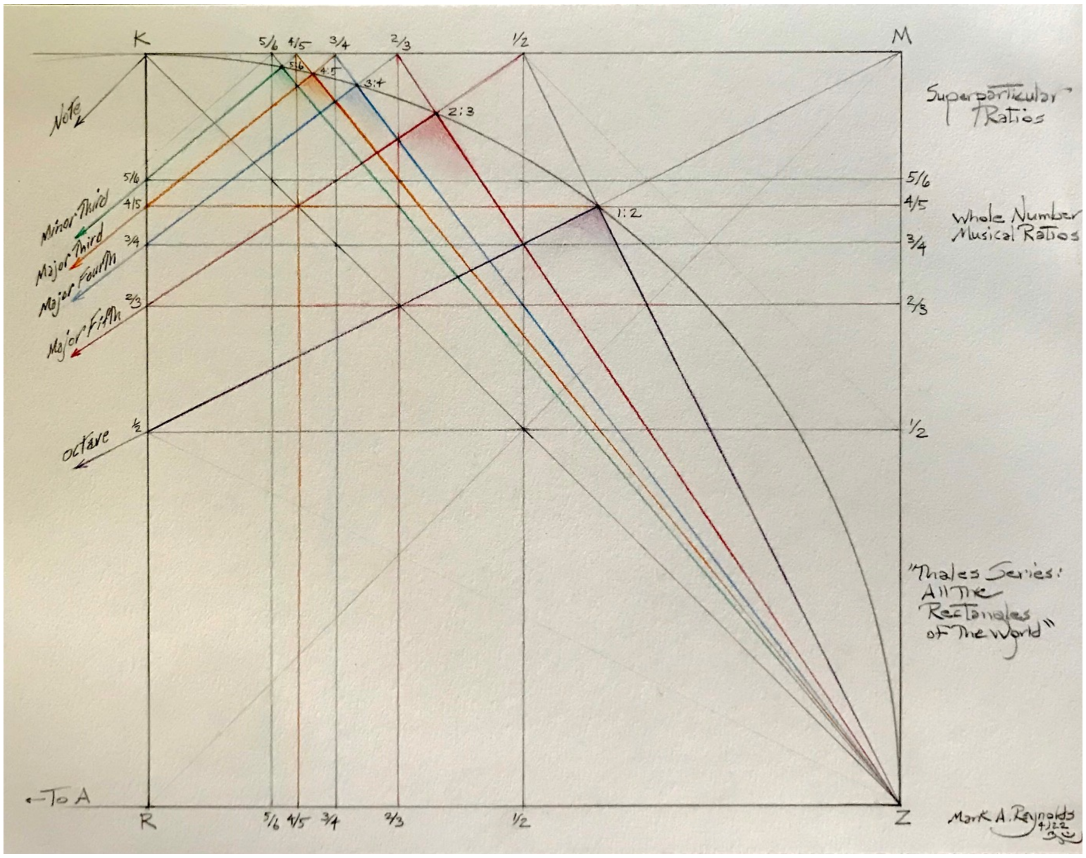


**Figure 2:** *"Superparticular Ratios: Whole Number Musical Ratios" from* Thales Series: All the Rectangles of the World *by Mark A. Reynolds; point R in the lower left-hand corner is* $(0,0)$ *in Figure 1.*

Ratio theory was one of the primary mathematical tools before calculus emerged, and scholars used names for ratios based on their characteristics (see, e.g., [13], which also helps to explain *why* Renaissance theorists such as Leon Battista Alberti and Daniele Barbaro used names for ratios). While some categories of ratios had quite convoluted names, others had rather compact names such as superparticular and superpartient. Superparticular ratios comprise the sequence of the form $(n+1):n$, so that the musical ratios cited above, $2:1,\ 3:2,\ 4:3$, are the first three superparticular ratios. In fact, the next two are the major third, $5:4$, and the minor third, $6:5$.

## Description and Verification of the Construction

We are now prepared for Reynolds' method, which inherently involves the Pythagorean triples $(a_n, b_n, c_n)$, to construct the superparticular family of rectangles with length-to-width ratios $(n{+}1):n$ for each $n = 1, 2, 3, \ldots$. Reynolds illustrates his construction in Figure 2, and we will need to analyze it in some detail. First note that Figure 2 is the first quadrant of Figure 1, as can be seen by matching the common label, the point $Z$. Since we are assuming the circle has radius 1, $Z$ is the point $(1,0)$, and $M$ in Figure 2 is $(1,1)$.

On the left side of Figure 2 one can see several arrows pointing towards the left and downward. They are all pointing to $(-1,0)$, the point $A$ in Figure 1, because the construction involves repeatedly drawing lines to $(-1,0)$ and intersecting them with the unit circle. In fact, the base step of Reynolds' construction, once the circle is drawn and $M$ is found, is to draw the line from $M$ to $A$ and intersect it with the unit circle. Lo and behold, the intersection point is $\left(\frac{a_1}{c_1}, \frac{b_1}{c_1}\right) = \left(\frac{3}{5}, \frac{4}{5}\right)$, and thus we see the first Pythagorean triple, $(3,4,5)$!

Note that the point $\left(\frac{3}{5}, \frac{4}{5}\right)$ is labeled "1 : 2" in Figure 2, since it is one vertex of a double square—think of $\left(\frac{3}{5}, \frac{4}{5}\right)$ as point $P$ in Figure 1.

Having located the first unit circle point corresponding to our Pythagorean triple family, $\left(\frac{a_1}{c_1}, \frac{b_1}{c_1}\right) = \left(\frac{3}{5}, \frac{4}{5}\right)$, how do we find the second, $\left(\frac{a_2}{c_2}, \frac{b_2}{c_2}\right)$? The answer is actually a two-step process, which can be roughly described as bouncing off of the line $y = 1$ and heading toward $(-1, 0)$. Specifically, referring to Figure 2, extend the line through $(1, 0)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$ to the horizontal line through $K$ and $M$. The intersection will be the point labeled "$\frac{1}{2}$" (because it is halfway from $M$ to $K$). Now take the line from this point to $(-1, 0)$, and intersect it with the unit circle. This results in $\left(\frac{a_2}{c_2}, \frac{b_2}{c_2}\right) = \left(\frac{5}{13}, \frac{12}{13}\right)$, which is the point labeled "2 : 3" (since it is one vertex of a major fifth rectangle). This two-step process can be repeated as desired.

Now that we have described Reynolds' construction, we can more readily contrast it to the well-known parametrization for all rational points on the unit circle described in the previous section. To find the point $W = (0, t)$ in Figure 1 for a given rational number $t = n/m$ would require either a ruler or some auxiliary straightedge-and-compass method. Reynolds' simple construction generates only a subset $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ of all rational points on the unit circle, but these are precisely those needed to find the superparticular ratios.

The remainder of this section comprises two theorems. The first shows that the two-step process above to get from $\left(\frac{3}{5}, \frac{4}{5}\right)$ to $\left(\frac{5}{13}, \frac{12}{13}\right)$ can be generalized to get from $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ to $\left(\frac{a_{n+1}}{c_{n+1}}, \frac{b_{n+1}}{c_{n+1}}\right)$. The second shows that each $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ is the vertex (labeled $P$ in Figure 1) of a superparticular rectangle with length-to-width ratio $(n + 1) : n$. Together these two theorems demonstrate the correctness of Reynolds' construction.

**Theorem 1.** *For each $n = 1, 2, 3, \ldots$, the line through $(1, 0)$ and $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ intersects the line $y = 1$ at $\left(\frac{1}{n+1}, 1\right)$. Moreover, the line through $\left(\frac{1}{n+1}, 1\right)$ and $(-1, 0)$ intersects the unit circle at $\left(\frac{a_{n+1}}{c_{n+1}}, \frac{b_{n+1}}{c_{n+1}}\right)$.*

*Proof.* The slope of the line through $(1, 0)$ and $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ is

$$m = \frac{2n^2 + 2n}{2n + 1 - (2n^2 + 2n + 1)} = -\frac{n + 1}{n}. \tag{2}$$

Thus the equation of the line through $(1, 0)$ and $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$ is $y = \frac{n+1}{n}(1 - x)$. Intersecting this line with the line $y = 1$ yields $x = \frac{1}{n+1}$, which completes the proof of the first statement.

The slope of the line through $\left(\frac{1}{n+1}, 1\right)$ and $(-1, 0)$ is

$$\tilde{m} = \frac{1}{\frac{1}{n+1} + 1} = \frac{n + 1}{n + 2}. \tag{3}$$

Thus the equation of the line through $\left(\frac{1}{n+1}, 1\right)$ and $(-1, 0)$ is $y = \frac{n+1}{n+2}(x + 1)$. Intersecting this line with the unit circle yields

$$\left(\left(\frac{n + 1}{n + 2}\right)^2 + 1\right) x^2 + 2\left(\frac{n + 1}{n + 2}\right)^2 x + \left(\left(\frac{n + 1}{n + 2}\right)^2 - 1\right) = 0. \tag{4}$$

Applying the quadratic formula cleans up quite nicely:

$$x = \frac{-\left(\frac{n+1}{n+2}\right)^2 \pm 1}{\left(\frac{n+1}{n+2}\right)^2 + 1}. \tag{5}$$

The negative root leads to $x = -1$, and then $y = 0$, which we already knew (the point $A$ in Figure 1). The interesting case is the positive root, which works out to be

$$x = \frac{2n+3}{2n^2+6n+5} = \frac{2(n+1)+1}{2(n+1)^2+2(n+1)+1} = \frac{a_{n+1}}{c_{n+1}}. \tag{6}$$

It remains to verify the corresponding $y$-coordinate, and for this it is easier to plug back into the unit circle, because then we can use the fact that $a_{n+1}^2 + b_{n+1}^2 = c_{n+1}^2$. We have

$$x^2 + y^2 = 1 \Longrightarrow \left(\frac{a_{n+1}}{c_{n+1}}\right)^2 + y^2 = 1 \tag{7}$$

$$\Longrightarrow c_{n+1}^2 y^2 = c_{n+1}^2 - a_{n+1}^2 = b_{n+1}^2 \tag{8}$$

$$\Longrightarrow y = \frac{b_{n+1}}{c_{n+1}}; \tag{9}$$

the positive root is taken in the last step since we are working in the first quadrant. □

**Theorem 2.** *For each $n = 1, 2, 3, \ldots$, if the point $P$ in Figure 1 is taken to be $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$, then the rectangle $APZT$ will have length-to-width ratio $(n+1) : n$.*

*Proof.* Taking the point $P$ in Figure 1 to be $\left(\frac{a_n}{c_n}, \frac{b_n}{c_n}\right)$, the length-to-width ratio of rectangle $APZT$ is

$$\frac{AP}{PZ} = \frac{AS}{PS} = \frac{1 + \frac{a_n}{c_n}}{\frac{b_n}{c_n}} = \frac{2n^2+4n+2}{2n^2+2n} = \frac{n+1}{n}. \tag{10}$$

□

## Geometric Art

While Reynolds is a devoted geometer, he is first and foremost an artist. Moreover, his geometric constructions provide the inspiration for his artwork. As such, it is only fitting to provide a few examples that were inspired by Thales' Theorem. The artwork shown here, Figures 3 and 4, are just two of the many in Reynolds' *Thales Series: All the Rectangles of the World*. Other works in this series, as well as a wide variety of his artwork in general, are available for viewing at https://markareynolds.com/.

## The $n$-Section of a Line Segment

As an added bonus, and as evidence that perhaps Reynolds' construction is indeed novel, we note that it also provides an easy solution to the problem of dividing a given line segment into $n$ equal parts. This problem, sometimes referred to as the $n$-section of a line segment, has a long and rich history, with solutions provided by such noteworthy historical figures as Euclid, Albrecht Dürer, and Niccolò Tartaglia [5]. Reynolds' construction provides the $n$-section ($n \geq 2$) for any line segment as follows.

Without loss of generality, take the given line segment to be $KM$ in Figure 2. Use the construction described above Theorem 1 to find the point labeled "1 : 2" in Figure 2, and then apply Theorem 1 itself $n-1$ times. This will result in the point $\left(\frac{1}{n}, 1\right)$ on $KM$, which is to say that it provides the $n$-section of $KM$. The fact that Reynolds' construction does not seem to appear in the literature on the $n$-section of a line segment provides some evidence that it might not have been observed before.

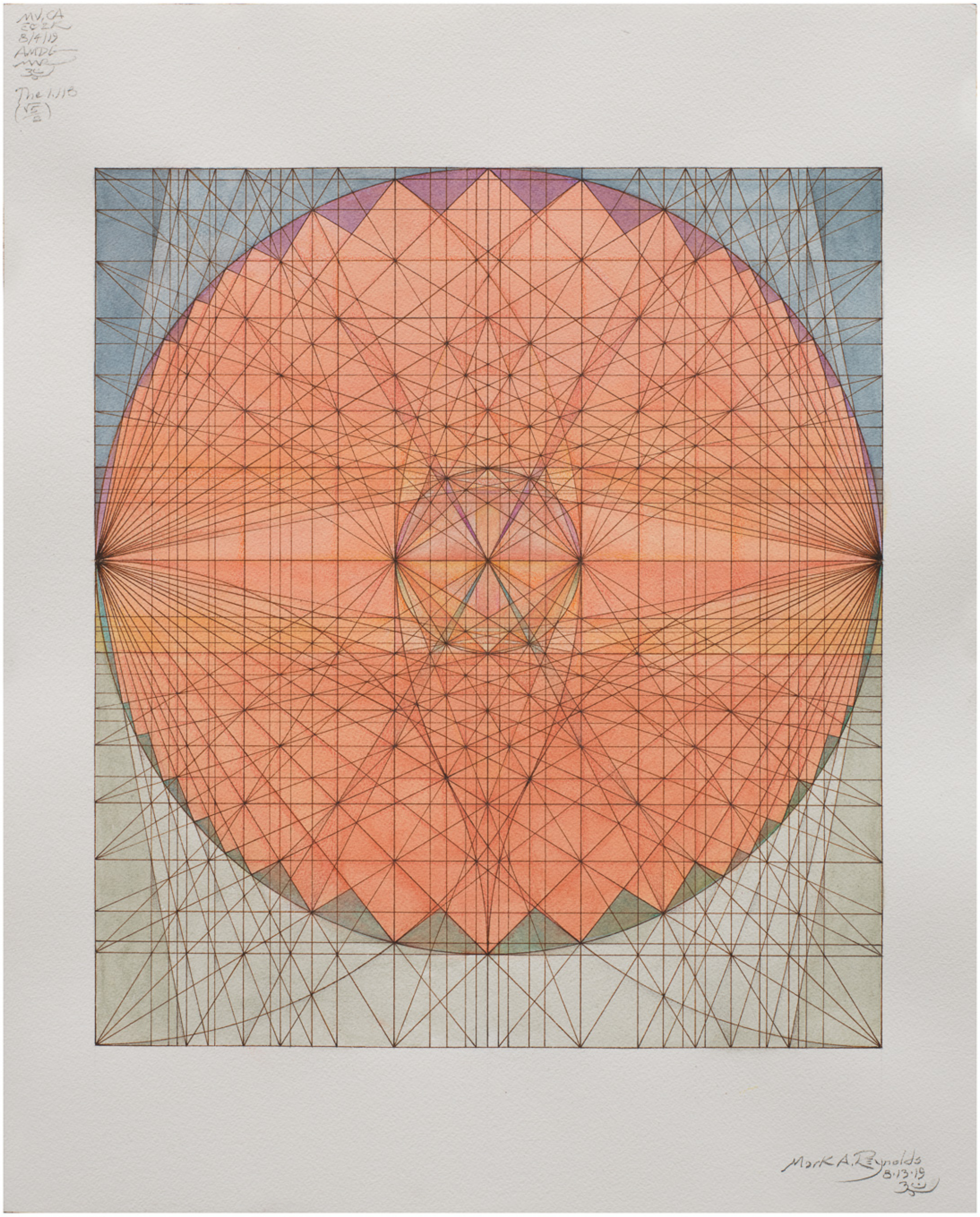


**Figure 3:** *Thales Series: ATROW, the 1.118, 8.12.19, 2019, 22 in. × 15 in.; watercolor, pen and ink on cotton paper.*

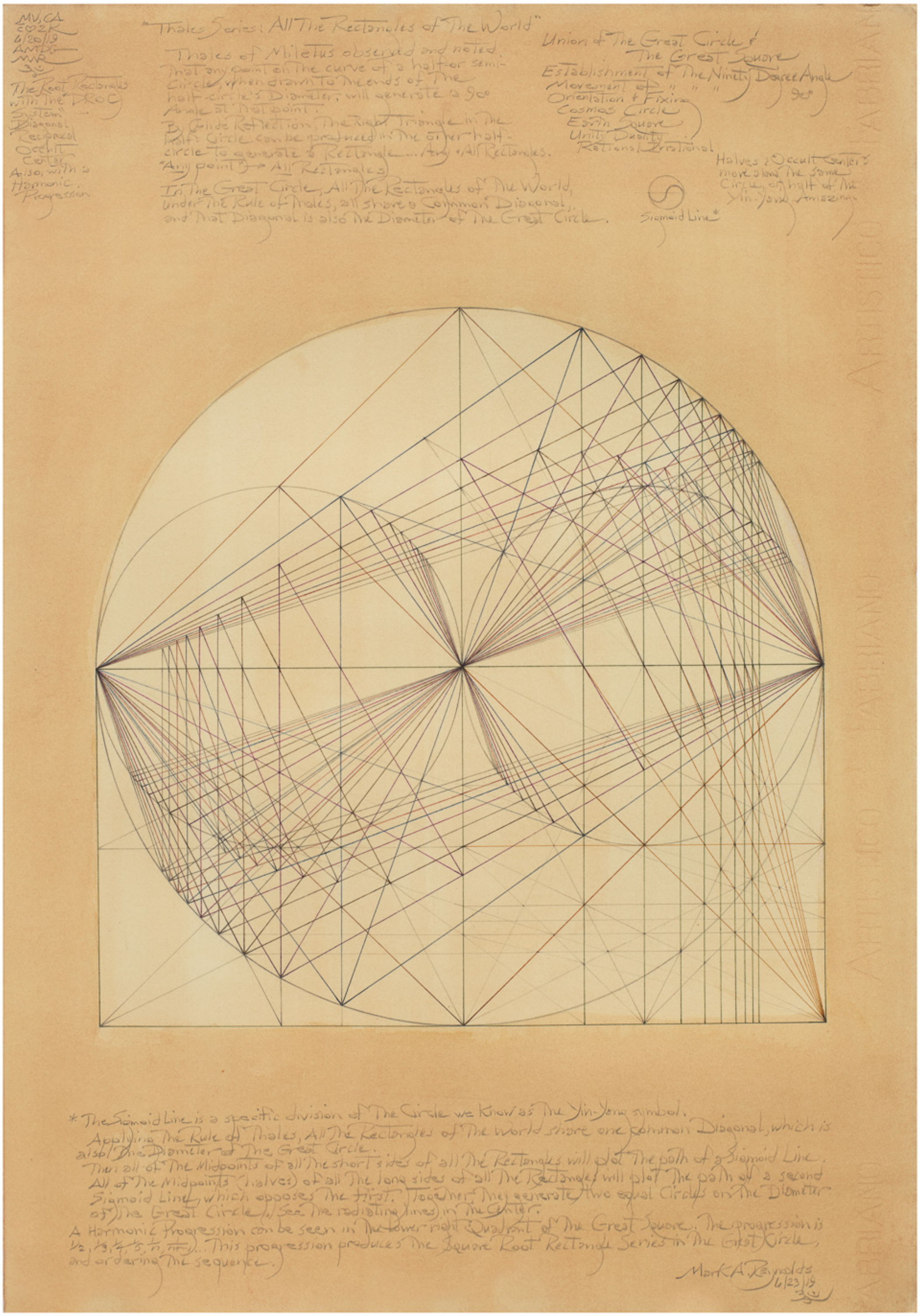


**Figure 4:** *Thales Series: ATROTW, Root Rectangles and the DROC System, 6.19, 2019, 22 in. × 15 in.; graphite and pen and ink on tea stained cotton paper.*

## Summary and Conclusions

We have presented and proven the correctness of Reynolds' geometric construction, inspired by Thales' Theorem, to produce a rectangle with any given length-to-width ratio of the form $(n+1) : n$ (for any positive integer $n$) inscribed within a circle. The construction inherently involves the well-known family of Pythagorean triples, $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, etc. We have also displayed a sampling of the artwork in Reynolds' *Thales Series: All the Rectangles of the World*.

While the appearance of the points $\left(\frac{3}{5}, \frac{4}{5}\right)$, $\left(\frac{5}{13}, \frac{12}{13}\right)$, $\left(\frac{7}{25}, \frac{24}{25}\right)$, etc., on the unit circle is well known, we have not been able to find anything in the literature relating these points to Reynolds' construction and to the superparticular ratios $(n+1) : n$ generated by his construction. Obviously, we would be most interested if the reader knows of any such source. Also, we do not claim that Reynolds' construction for the superparticular rectangles is the simplest nor that our proof of correctness is the simplest possible (though they seem quite simple and elegant to us!). The point is that the construction was discovered by Reynolds while he was on the geometric/artistic quest that resulted in his *Thales Series*.

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