# **Juggling Dancers: Passing Props and Partnering Paths**

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#### Abstract

Prop and object manipulation by dancers bears resemblance not only to juggling but also to the ways that performers in duet or trio dance forms often adroitly switch partners, and we examine examples of both. "Juggling" here is interpreted broadly to refer to all forms of prop manipulation, as well as to the fact that dancers are also "juggled" from one partner to another. We analyze the signature work of choreographer Charles Moulton, "Precision Ball Passing," which involves movement choruses of large numbers of dancers virtuosically passing balls back and forth in a manner demanding cooperation and community. We also examine how geometry and combinatorics are often intrinsically embedded in such dance works and discuss several examples.

# Introduction

In many performing arts forms the performers routinely handle and manipulate props or objects, often set to music, and performed with expressiveness and virtuosity, and these performances often include recognizable elements of dance. The exchange or passing of objects may show resemblance to the switching of partners in dance works, and we will point out some of those connections. We will examine in more detail one very famous object passing dance, "Precision Ball Passing," the signature work of choreographer Charles Moulton. Mathematical thinking and design are often hidden "in plain sight" within choreographic works, and choreographers often chunk together simple, easily repeatable patterns in time, space, and sequence to create enjoyable visual experiences for audiences. Choreographic simplicity may make performance of a dance more accessible to a wide range of performers, yet still create complex imagery.

Prop manipulation has been one of the hallmarks of the dance company co-directed by the author, the Dr. Schaffer and Mr. Stern Dance Ensemble, operating under the auspices of MoveSpeakSpin. Props that we have created dances with include oversized tangrams and similar polygonal puzzle pieces, sections of PVC pipe, ordinary sheets of paper, giant and medium sized flexatubes, rope, string, and bungee cord loops, 5 gallon and smaller water bottles, basketballs, gatorboard cubic boxes, kitchen knives, and many more.

Work with props has led us to a number of insights about how best to work with them in dance, as described in more detail in [12]. Ordinary mathematics manipulatives enlarged to giant size almost always generate performance as soon as people of any ages begin playing with them. But skill with props is usually not immediate, and careful rehearsal is necessary for them to fluidly become a genuine and virtuosic extension of the moving body and a usable creative element in performance. Performance with props is central to sports, circus arts, gymnastics, and many other human activities. The author learned much about prop manipulation from two performers with honed skills, circus artist and clown Rock Lerum who can twirl just about anything you throw at him, and our dance company member Saki who often easily blends her circus and rhythmic gymnastics background into dance work. Over the years, Erik Stern and I spent endless hours rehearsing with props to incorporate them into dances, and often found that finding surprising ways to perform with everyday objects can create enjoyably charming effects on stage.

Oversize math manipulatives typically give rise to new mathematical problem solving not found when the smaller hand-sized objects are handled. For example, in collaborative work, one person cannot hog and control the objects, and so who holds which and in which hands usually adds complexity that requires close attention and real cooperation. Switching hands or switching who holds which prop often becomes a geometric and combinatoric problem. If the props are used to create shapes or designs, then moving into, out of, and from one design to another become an added geometric and movement puzzle.



**Figure 1:** (*a*) Four vertical positions of the double-tans. (*b*) Possible closed positions. (*c*) One kaleidoscope design. (*d*) to (*e*) Example of transition to ending position.

For example, in a recent quartet [14] each performer danced with a "double tan," two same sized right isosceles triangles painted fluorescent yellow on one side and fluorescent red on the other. They are joined together at one vertex around which can be rotated only within the plane of the shapes, Figure 1. The double tans are built from several inch thick insulation foam, and the small circles are knobs that make handholds easier. This work was created as one section of a performance of Terry Riley's early minimalist piece "In C," performed on Nov. 13, 2021, in a park by 25 musicians of the Santa Cruz, CA orchestra New Music Works, and including dance work by 7 choreographers and over 25 dancers. Our double tans section is a work in progress that begins and ends with the dancers marching on and off stage with the double tans resting on their backs, suggesting the image of dinosaurs, Figure 1e. Towards the end of the piece the dancers create a fluid sequence of pretty kaleidoscopic images, the last of which is Figure 1c. Ultimately we will use a different score, and the dance will be an environmental work humorously suggesting that perhaps we, somewhat like hypothetical dinosaurs, are creating a magnificent civilization before marching into the eternal sunset of environmental collapse – in the spirit of a pointedly satiric video created by the United Nations in 2021, [16].

Note that there are four positions in which the double tan can be held in vertical position with the triangles' surfaces facing the audience, Figure 1a. Since either the right or left hand might hold the upper triangle's handle, there are eight simple ways that one dancer can hold one set vertically. This becomes slightly more complicated because sometimes the dancers need to use two hands, one to hold each of the two knobs, or sometimes to grasp the edges of the shapes. The double tans rotate fully to form either a square or a larger right isosceles triangle, Figure 1b, and these can be brought together in vertical kaleidoscopic patterns displayed to the audience by the dancers, Figure 1c.

At the end of the dance the four performers had to move from their positions in Figure 1c through at least one of the eight vertical positions, such as that in Figure 1d, to that in Figure 1e, so finally all yellow sides faced the audience along the dancers' backs, as the dancers marched toward stage right (the reader's left) Note that in Figure 1c Jane is holding her piece by reaching upwards, Laurel downwards, and the other two either right or left. Each dancer had to individually do the geometric and manipulative problem solving in rehearsal on how to accomplish the transition smoothly and efficiently. This transition took only a few seconds in a  $6\frac{1}{2}$ -minute dance. Such geometric puzzle solving is a constant necessity in performance work, especially when props are used and involved in transitions from one design, image, or movement motif to another. The video clip [14] from the dance also includes a short phrase in which symmetric designs smoothly morph, which again requires careful attention to geometric properties and manipulation techniques. Again, this is to point out that when manipulatives are greatly enlarged, new mathematical problem solving arises quite naturally.

# **Passing Props**

In 1990 Erik Stern and I generated a collaborative classroom activity involving counting numbers of possible handshakes as the outgrowth of a vaudevillian duet routine in which we at first try unsuccessfully

to shake hands, then get our hands stuck together [11, 12 Ch. 1, 13]. The classroom activity essentially takes a topological point of view about what constitutes distinct handshakes, as two handshakes are considered different if different hands connect with each other, rather than if the shapes of the arms and bodies or the hand grasps are different during the handshake. Even so a variety of answers are usually given by participants. For example, if person A joins right hand to person B's left hand that would be considered different than person A's left to person B's right – unless one does not consider people as distinct, in which case there is one possibility of right to left. Erik and I long ago decided that the goal of the exercise was not so much to get students to give "correct" answers, but to encourage discussion in which participants clarify and defend their count as hopefully consistent with their interpretation of the rules. For a number of years, I have extended this activity by having small groups of participants in workshops pass small objects from hand to hand when they engage in the handshake activity, while they explore how combinations of passes generate interesting counting and permutation group problems. We will see that activities like this resemble how dancers switch partners in certain dances, but also how juggling patterns and dances involve passes of objects.



**Figure 2:** (*a*) *Example of two trio handshakes with object passing.* (*b,c,d*) *Handshakes with 3, 1, or 0 selfpassings respectively.* (*e-i*) *Two trio handshakes with object passing of periods 6,6,2,6,4.* 

One exercise has workshop participants work in trios to create a sequence of several three-person handshakes, incorporate dance elements such as level, rhythm, and dynamic changes, practice and perform them, and then discuss the performances. Another exercise asks them to create two handshakes in which each person in the trio uses both right and left hands as part of each handshake. Each participant is given one small object like a ball and is asked to hand off the object when the hand holding the object shakes with an empty hand. The second handshake must then similarly pass the objects to empty hands. See the example in Figure 2a, in which the objects are a circle, square, and triangle. The solid arrows represent the first set of handshakes. For example, the second dancer hands the circle from the left hand to the first dancer's left hand. The third dancer hands the triangle from right to left hand, which we might call a "selfie" handshakes each object returns to its original hand, so this set of handshakes has period six. Workshop participants are asked to keep track of how many handshakes it takes for all objects to return to their starting hands, and to try to find two handshakes that combine so that each object's "orbit" includes all six hands.

Note that once the starting hands of the three objects are specified, after an odd number of the two sets of handshakes the objects will end up in the other three hands, and vice versa after an even number of handshakes. Therefore, the period in this problem – if it exists - will always be an even number. In fact, it must exist and the paths the objects follow will consist of cycles, including possibly "degenerate" cycles of length two. That is because each hand, considered as a vertex in the overall directed graph, such as Figure 2a, has one incoming and one outgoing arrow edge. The period will be the least common multiple of all these cycles' lengths. This of course will remain true if a larger number n of dancers engages in the same exercise, involving n objects and two similarly constructed group handshakes.

We can count and categorize types of handshake handoffs in this scenario as follows. If each person in the trio starts with a distinct object in their right hand, then if we allow "selfies" or self-handoffs there are 3! = 6 ways to simultaneously distribute those three objects to the three left hands (or 2! = 2 if selfies are not allowed). However, each person might have decided in one of two ways which hand will start with the object and which receive, so the actual number is  $2^3 \cdot 6 = 48$ . If self-handoffs are disallowed, the number is  $2^3 \cdot 2 = 16$ . Again, we are assuming that people and objects are distinct. The second set of handshakes and accompanying object handoffs are more limited. For example, if after the first round the objects are in left hands and only right hands are next available to receive, there are 3! (or 2! if no selfies allowed) handshakes possible in round two. Thus, the total number of two handshake sequences is surprisingly large,  $2^3 \cdot 6 \cdot 3! =$ 288, or  $2^3 \cdot 2 \cdot 2 = 32$  if no selfies. In general, for *n* people, the numbers are  $2^n \cdot n! \cdot n!$  or  $2^{n-1} \cdot (n-1)! \cdot (n-1)!$ 

We might wish to categorize trio handshakes according to numbers of self-handoffs. Figure 2b,c,d show the only possible types of trio handoffs possible: 3-selfies, 1-selfie, and 0-selfies respectively. We might also wish to tabulate the pairs of handshake sets according to their cycle lengths. Figure 2e,f,g,h,i show several examples pairs of trio handoffs incorporating combinations of the number of self-handoffs:

- (A) Figure 2e: 3-selfie and 0-selfie, period = 6 (B) Fig
  - (B) Figure 2f: 3-selfie and 3-selfie, period = 6(D) Figure 2h: 1-selfie and 1-selfie, period = 6
- (C) Figure 2g: 3-selfie and 3-selfie, period = 2(E) Figure 2i: 1-selfie and 0-selfie, period = 4

To complete the categorization, not shown in Figure 2 are examples of 1-selfie and 0-selfie, period 2; 3-selfie and 0-selfie, period 4: and 3-selfie and 3-selfie, period 2. This kind of analysis can emerge during the workshop activity described above, but we will see how examples of these patterns also arise in an important choreographic work. These patterns are very similar to - though certainly physically easier than - actual three-person juggling patterns [17, pp 46-55], and even show resemblance to passing patterns in sports such as basketball [15].

### **Precision Ball Passing**

Charles Moulton is a well-known choreographer whose signature work, "Precision Ball Passing," involves dancers sitting and standing in a rectangular array in bleacher seats executing complex sequences in which balls are handed off from performer to performer. Originally created for three dancers in 1979 within the era of the postmodern dance movement, the work has since been performed by larger ensembles of sizes 9, 18, 25, 36, 48, 60 and 72 [10]. The critic Jack Anderson has written, "Because such compositions are based upon such a banal activity as ball tossing, they could be called parodies of all types of virtuosity in dance. At the same time, their very complexity makes them virtuosic in their own right."[1] Although here we will examine some of the mathematical elements in the complex sequences of the dance, Moulton has talked about the importance of the work's collaborative, problem-solving, community building, and transformative aspects: "Ball passing is both a game and a dance. It's a human puzzle that groups can only solve by working closely and cooperatively together" [10]. When he taught it to a group of children undergoing difficult medical treatments at a hospital in Germany, he said,

"They invented new ways of passing and receiving, quickly adapting to the problems of ball passing, and coming up with surprising solutions. No matter what physical issues these kids were facing, their creative skills were vibrant, alive, and healthy. Their ability to play, and through play, to solve complex problems, was very strong. In the hospital they experience themselves primarily as patients, but in the environment of ball passing they were empowered to become leaders, creators, and problem-solvers." [9]

We will examine several sequences in a public online video of a 7-minute performance of "Nine Person Ball Passing" [8]. The analysis demonstrates that many patterns in the dance are performed by trios using designs such as those outlined above. The piece overwhelmingly involves controlled handoffs of the balls, though a few actual throws of the balls into the air as in juggling are "tossed" in here and there. However, the piece could be considered a juggling dance in which most passes take one and sometimes two beats, performed to a mostly 8-beat and occasionally 6-beat musical score. I believe the graph-theoretic

depiction outlined above and shown in Figure 2 is more helpful in understanding the mathematical elements of this work than site swap juggling notation used in analyses of juggling patterns in which tossed balls often stay in the air for different numbers of beats.

When watching the dance, observe that the central dancer 5 has a very complex role, and the dancers frequently – and cheerfully – seem to be giving verbal cues to each other. Moulton has used many variations on the patterns in this work in his larger ensemble ball passing pieces, though trio patterns like those above seem to be basic in all of them. Overall, these are quite beautiful and virtuosic dance works composed of simple passing movements performed by dancers remaining in place.



**Figure 3:** (*a-e*) Moulton's trio and duet ball passing patterns. (*f-j*) 8-beat phrase in which 1 or 2 balls rather than 3 are passed.

The dance begins with performers 4, 5, and 6, the middle trio in Figure 3, executing 16 counts of the Figure 2e period 6 pattern. They then switch to the mirror image 16 beat pattern with the left hands rather than the right hands passing the balls to the next dancer. Because the music is in 8-beat phrases there is not a sense in which the balls return to their starting points in time to the musical phrases – this is not difficult to track because this trio uses red, yellow, and green balls. After 8 more beats of the initial right hand passing pattern, the dancers execute 8 beats of the "back and forth" period 2 pattern in Figure 2g, but at half speed. Similar patterns continue, with occasional gestures with the balls towards the audience. These sometimes complete a period 6 passing pattern with 2 more beats for the forward gesture. At 0:46 these three dancers execute a period 2 type I/I selfie/selfie pattern high in the air at half speed for 8 beats. At 0:54 the other six dancers join in, dancers 1,2, and 3 forming a bottom row trio, and 7, 8, and 9 porting a top row trio, using the Figure 2e patterns like the first trio's phrases. Meanwhile dancers 4, 5, and 6 perform a sequence not outlined above, but shown in Figure 3 f-j (distinct shapes rather than circles are shown for clarity), in which either 1 or 2 balls are passed on each movement, rather than all 3 balls. It appears that each pass takes 2 beats, and the pattern repeats after an 8-beat phrase. The way the dancers lean in the same direction during the passes accentuates the effect.

As the dance proceeds the middle trio continues the previous passing patterns, but with the arms beginning to execute more complex interweaving figures, each pass now taking 8 beats, and the passes alternating the passing hands first right, then left, etc. At 1:28 the middle trio executes the same pass pattern, but now with arm formations that appear knotted, though they are not. At 1:39 the bottom trio continues the same patterns, but 4 and 7, 5 and 8, and 6 and 9 form 3 duets, exchanging balls vertically. We will not break down the dance further, but Figure 3a-e show the various arrangements of trios and duets used throughout: (a) three horizontal trios, (b) three vertical trios, (c) three vertical duets and one horizontal trio (there are three variations of this pattern), (d) two corner trios and one diagonal trio (two variations), and (e) three triangular trios (two variations). The vertical and diagonal trios are of the form Figure 3f-j, and the patterns used in the Figure 3e formation are the Figure 2e period 6 pattern.

#### **Passing Dancers**

Interestingly we can find similarities between how dancers might pass objects around and how dancers can pass partners back and forth. In what follows we will look at some possibilities. We can turn the previous analysis of ball passing trios into a description of a dance sequence for six dancers. Three of the dancers, say wearing blue costumes, A<sub>1</sub>, A<sub>2</sub>, and A<sub>3</sub> stay in place while the other three wearing red costumes, B<sub>1</sub>, B<sub>2</sub>, and B<sub>3</sub> move in a cycle among the As, partnering with each A<sub>i</sub> in turn. Here the Bs are like the passed balls, though we are not differentiating positions right and left within the duets, as with the ball passing. In general, for *n* stationary dancers A<sub>i</sub> and *n* rotating dancers B<sub>j</sub>, there are (n - 1)! possible "Hamiltonian cycles" which allow each A<sub>i</sub> to partner with each B<sub>j</sub> in the sequence. For n = 3 there are only two such cycles, for n = 4 there are (4 - 1)! = 6 cycles, the three shown in Figure 4(a) and those three cycles with their arrows reversed. Several authors have examined symmetry transformations among 4 or more sets of square dance partners, for example [1,4].



**Figure 4:** (a) Hamiltonian cycles for four duet pairs. (b) Adaptation of Figure 2(a) for duets by 6 dancers, red costumed dancing with blue. (c) Inner and out circles in partner mixer dance.

We might also adapt Figure 2(a) for six dancers in two sets of three, even numbers costumed blue and odd numbers red, who each are to perform as part of a duet in every one of six positions on stage, as in Figure 4(b). The overall path is the same as in Figure 2(a), and in this case the six dancers all rotate along the six-cycle, and each dancer does perform with each oppositely costumed dancer. If a simpler design such as Figure 2(e) is used, then each dancer will only partner with the oppositely costumed dancer before or after them within the cycle. The constraint that dancers are in two sets with equal numbers, and each dancer partners with each dancer in the opposite set is a popular form of folk-dance mixers, such as the "Paul Jones Mixer" [4]. Contra dances often have each couple dance with each other couple, and also in each location [5]. In "Down by the Corner" [7], Figure 4(c), the inner circle of dancers moves forward counterclockwise one position, while the outer circle remains in place. Other folk-dance mixers use more complex rules, for example the dance "Bingo," [3], in which (traditionally) the women circle CW and men CCW, passing their own partner then the next three in line, and calling out one letter of B-I-N-G-O per opposite dancer, before reaching their next partner on the letter O. The dancers move forward four spaces, so this will only result in each dancer eventually partnering with each opposite group member if the total number in each circle is relatively prime to four, that is if the numbers are odd.

We might ask that for two groups of red and blue dancers, each of size n, each dancer partner once with each opposite group member for one round of an n round dance, but also that each dancer appear once in each of n designated positions on stage. This criterion has been investigated for square dance formations of four couples in [2], in which the authors explain the connections to pairs of Graeco-Latin or orthogonal Latin squares. An order n Latin square is one in which each value 1,2,3,...,n occurs once in each row and once in each column; two such squares are orthogonal if when the squares are superimposed each pair of values from  $\{1,2,3,...,n\}$  occurs once as an ordered pair among the  $n^2$  positions. Here I will only note some of the rules dancers must follow to accomplish this, which are simple for odd n, more complex if n is even, and impossible for n = 2 or 6, since no pairs of such orthogonal squares exist for those *n*. In Figure 5 (a,b) is shown one pattern for two groups of five dancers derived from the pair of orthogonal Latin squares indicated by the sets of first and second coordinates in the Figure 5(b) grid. Each outer blue dancer rotates one position CCW and each inner red dancer rotates one position CW per round. This well-known pattern easily generalizes for all odd  $n \ge 3$ . Also shown in Figure 5 (c,d) is a pattern for two sets of four dancers; the switches of the dancers are more complicated. Even number patterns for  $n \ne 2,6$  are more complex than for odd numbers, but the existence of such Graeco-Latin squares was a long-time unsolved problem, now solved.



**Figure 5:** (*a*,*b*) Orthogonal Latin square pattern for circle of 5 pairs. (*c*,*d*) Orthogonal Latin square pattern for four partner pairs, arrows show moves from first to second round.

**Pair Sharing for Dancers**. Suppose that instead of two sets of six dancers who must efficiently partner all 36 pairs from the two groups, each dancer appearing in six locations on stage – which we now know to be impossible - twelve dancers are performing a piece in which each dancer is meant to partner with each other dancer as efficiently in terms of time and geometric spacing on stage as possible. This is a problem that also arises in conference or professional development settings in which it is desired that each participant spends several minutes in short discussion with each other participant, a procedure known as "pair sharing." For twelve dancers we can represent the problem as finding a "perfect matching decomposition" of the  $\binom{12}{2}$  =  $\frac{12 \cdot 11}{2}$  = 66 edges of the complete graph K<sub>12</sub> that specifies eleven sets of six pairs of mutually disjoint edges



**Figure 6:** (a) Complete graph  $K_{12}$  on 12 vertices. (b)Turning method of decomposing  $K_{12}$  into 11 sets of six disjoint pairs. (c) Adaptation for 12 dancers. (d) Similar decomposition for odd numbers of dancers.

without duplication using a technique known as "turning", Figure 6 (a,b), [6, pp 32-34]. The idea is that one dancer, in this case number 12, stands apart and rotates around the circle in the figure, sequentially partnering with dancers 1 through 11. Meanwhile the other dancers each partner with the dancer directly across the line linking 12 and 12's partner. In each of the 11 sets these five opposite partnerings (or edges) are parallel, never duplicate, and along with those of 12 exhaust the full set of edges. This pattern easily

generalizes for any even number of dancers. Figure 6 (c) shows that if instead of 12 rotating, the dancers each rotate one position counterclockwise, the same partnerings are accomplished, and the audience, here indicated to the dancers' right, has a good view of all partners. (For pair sharing, the partners might sit across a long table, or be assigned to one of six breakout rooms if done with online software.) For an odd number of dancers, each dancer simply takes a turn sitting out (or dancing a solo), as in Figure 6(d).

## **Summary and Conclusions**

This paper explored several instances of prop manipulation and partner switching in dance, including some of the geometric and combinatoric connections between them. We analyzed some of the mathematical elements found in Charles Moulton's iconic dance work "Precision Ball Passing." Partnering in dance and prop handling in performing arts are complex and embody mathematical concepts in rich and sometimes surprising ways. There is much more to be discovered and investigated in how this plays out in dance.

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