Crested Cactuses and Mathematical Sculpture

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Abstract

I describe features evidenced in cactus forms that are related to phyllotactic spirals, fractals, and tiling. I examine in greater depth plants exhibiting the growth abnormality known as cresting, in which superlinear growth creates structure related to hyperbolic geometry and fractal curves. I show five recent abstract clay sculptures inspired by crested cactus forms and describe the approach used for building each one.

Cactus Morphology and Mathematics

Fractal geometry and chaos, tiling and polyhedra, and phyllotactic (Fibonacci number) spirals are all evidenced in cactus forms. In addition, superlinear growth and hyperbolic geometry are exhibited by crested cactuses. In order to understand the role of mathematics in these forms, some basic understanding of cactus morphology will be helpful. For more details on cactus morphology please see Reference 1.

Cactuses assume a variety of overall forms, as shown in Figure 1. These include globular plants such as barrel and pincushion cactuses, columnar plants such as saguaros and organ pipe cactuses, branched plants with either cylindrical (e.g., chollas) or paddle-like (prickly pears) segments. Many cactuses exhibit branching, but the degree of branching varies greatly over different species [1].

Figure 1: *Cactuses with a) globular, b) columnar, c) cylindrical branched, and d) paddle-like branched morphologies.*

The branch-like features in cactuses are called stems, which are often ribbed to allow expansion when water is taken in through the roots. In some species stems are covered by protuberances called tubercles (e.g., Figure 1(c)). Cactus spines, which are highly modified leaves, are produced from areoles, a kind of highly reduced branch with a roughly hemispherical shape (Figure 2).

Figure 2: *Areoles and spines of saguaro and Baja organ pipe cactuses.*

There are two visually interesting growth abnormalities found in many cactus species, cresting and monstrosity [2]. These conditions result from a combination of genetic defects and damage to the growth front (the apical meristem). They can radically alter the morphology of a plant, often resulting in fantastical forms with, particularly in the case of crested specimens, considerable sculptural beauty (Figure 3). In cresting, growth at a point is replaced by growth along a line, while in monstrosity, growth occurs at numerous uncoordinated points, often leading to lumpy, asymmetric forms [2]. In normal columnar cactuses, additional ribs are occasionally introduced as the plant grows larger. An example is shown in Figure 3(b), where the appearance is seen to be similar to that of a dislocation in a crystal lattice [3]. In crested (aka cristate) forms there are many such "dislocations" to accommodate superlinear growth.

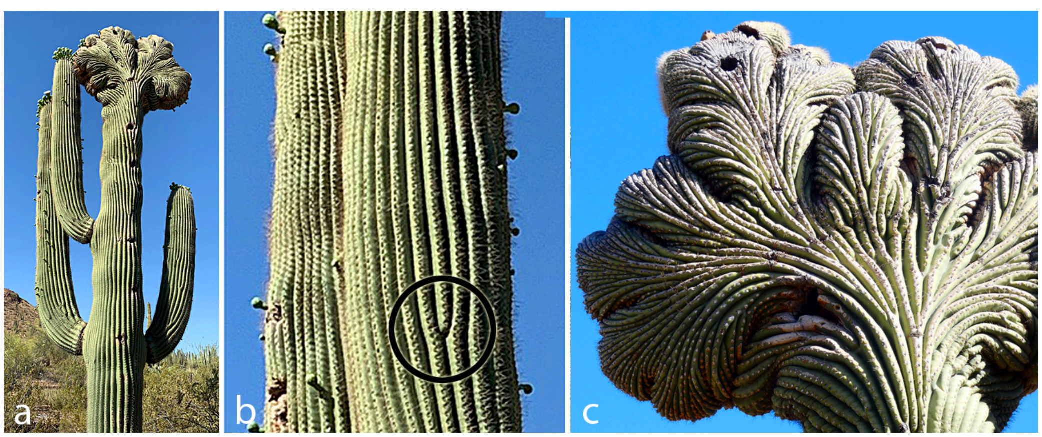

Figure 3: *a) A crested saguaro; b) close up of a "dislocation"; c) detail of the crested region.*

Fractal structure is evident in cactus branching as in other branched plant forms. This branching is chaotic, with small differences during development of the plant, likely including environmental factors, leading to a considerable variety of forms based on the same basic genetic instructions. Branching is also seen in roots of cactuses, as in other plants.

In some types of cactuses, the surface of the plant is essentially tiled by tubercles. For cylindrical sections, these patterns can be thought of as a planar tessellation wrapped around a cylinder. In a cactus, an edge-to-edge tessellation of parallelograms is a decent approximation to observed patterns, as seen in Figure 4(a). Rounded ends of stems can be looked at either as tessellations of spheres or polyhedra (Figure 4(b)).

In addition, the distribution of thorns on an areole is related to polyhedra through the problem of evenly distributing points on a sphere (Figure 2). In the case of a cactus areole, however, there is often a large central spine, with smaller spines near the perimeter.

Figure 4: *Tessellations of tubercles in a) a cylindrical region of a cholla cactus and b) the rounded end of a pincushion cactus.*

The distribution of areoles in cactuses generally follows Fibonacci-number spirals. An example is shown in Figure 5(a), where spirals connecting areoles are drawn over a photograph of a pincushion cactus. In this case there are eight spirals rotating in one direction and 13 in the other. An unfired clay sculpture with these same numbers is shown in Figure 5(b), where a custom-made clay tool was used to impress curved grooves on a smooth, hollow globular clay form. This bears some similarity to pineapple or pinecone decorative sculptural elements such as finials, often made of wood. These generally exhibit mirror symmetry, with the same number of spirals in each direction, in which case they do not accurately reflect the Fibonacci spirals seen in nature [4].

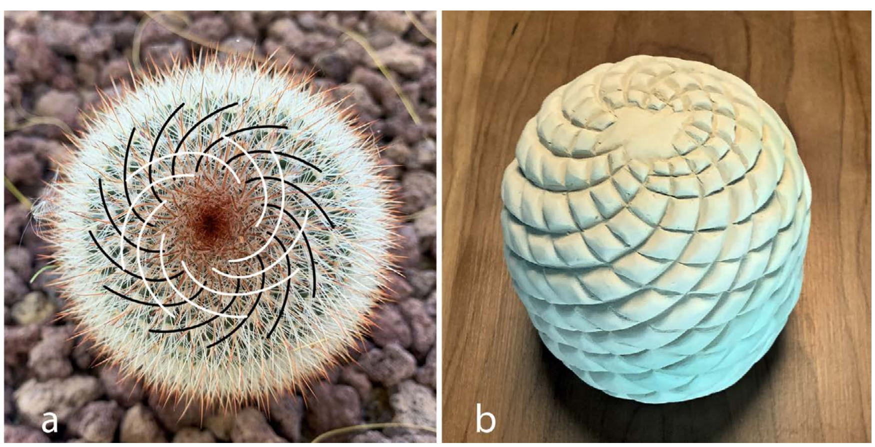

Figure 5: *a) Phyllotactic spirals in an Arizona Fishhook cactus. b) Clay sculpture using the same Fibonacci numbers, 8 and 13 (2022, unfired).*

Crested Cactuses

The morphological changes in cactuses due to cresting vary greatly with cactus type. While the term "exponential growth" is often used for wavy surfaces caused by superlinear growth, such growth isn't necessarily exponential. For this reason, "superlinear growth" is used here. Growth in crested cactuses along a linear front can lead to fan-like forms, undulate forms, and in some cases heavily-folded forms reminiscent of fractal curves. Globular cactuses like barrels and pincushions assume a densely folded aspect similar to that of brain coral [5] or hyperbolic crochet [6], as seen in Figure 6. In these plants the density of the folded structures is so high it's difficult to see what's going on in the interior, particularly with a corresponding high density of spines obscuring one's view. Columnar cactuses such as saguaros and blue myrtles and branched cactuses such as jumping chollas generally exhibit localized fanlike forms. In columnar cactuses the fanned forms are facilitated by the introduction of numerous additional ribs, or "dislocations", as seen in Figures 3(c) and 7(a). In chollas, which have pronounced tubercles, fanning is accommodated with additional rows of tubercles (Figure 8).

Also notable is the distorted form of the "bunny ears" prickly pear (regular form in Figure 1(d)). The paddles in these plants undulate as a result of superlinear growth, as seen in Figure 7(b). This plant has been described both as crested and monstrose [2, 7] but does not appear to exhibit growth along a line, which appears to be a defining characteristic of crested plants.

Figure 6: *a) Crested golden barrel cactus b) with the approximate outwardly-visible growth front delineated.*

Figure 7: a) *Fanned forms in a large crested blue myrtle cactus, showing how additional ribs accommodate superlinear growth. b) Monstrose "bunny ears" prickly pear cactus. Both photos were taken at the Desert Botanical Garden in Phoenix, Arizona.*

While growth in normal cactuses occurs at single points at the ends of branches, growth in crested cactuses occurs along lines. This disrupts the Fibonacci-number spiral patterns normally observed. In Figure 8, the pattern in a crested arm of a jumping cholla is delineated. It's possible to a degree to trace curved rows of tubercles in the crested arm (though breaks and disordered regions are unavoidable), but the distribution is too chaotic near the growth front to say anything about numbers of rows. This sort of disorder is seen near the center of Fibonacci-number spiral growth from a point as well, but it is generally straightforward to count spirals a little distance out from the center.

Figure 8: a) *Fanned form in a crested jumping cholla, b) with growth curves delineated.*

Sculpture Inspired by Crested Cactuses

Crested cactus forms were the inspiration for several recent ceramic sculptures. These are intended to be abstract pieces whose form is based on crested cactuses rather than realistic representations of cactuses. No attempt was made to form realistic areoles or to include spines. In some of them even the overall form is not closely related to an actual cactus.

Hyperbolic surfaces with ridged structures and "dislocations" such as those of Figures 3 and 7(a) above were the inspiration for two recent pieces. In the first, entitled "Groovy Hyperbolic Form," I started by forming a hyperbolic surface and then pressed ridges into it using a clay tool made for that purpose (Figure 9). The ridge-forming process was relatively easy to carry out, but the resulting piece does not fully capture the sculptural quality of a crested cactus. In the second piece, entitled "Crested Cactus Form," I started by building a three-lobed surface, with some hyperbolic character given to each lobe. I cut strips with a triangular cross section and then affixed them one by one, as shown in Figure 10(a). This was a relatively laborious process, and considerable care was required at the edges of the paddles where strips from the two sides meet in zipper-like fashion. The resultant sculpture, shown in Figure 10b, better captures the complex structure in these plants.

In crested cactuses, growth is observed to take place along a line. While there are a number of articles on the botany of cristate growth (e.g., Reference 8), I'm not aware of any mathematical models. In contrast, phyllotaxy from a point has been modeled for over a century [9]. Since plants are composed of cells, a continuous line is clearly not an accurate model. A reasonable supposition, therefore, is that growth proceeds from a series of closely-space points. These can be readily modeled using phyllotactic spirals. In Figure 11(a), three such spirals are spaced along a line. An attempt was made to graphically extend the spiral arms in such a way that they interact to form an array of curves similar to those observed in Figure 8. Note that extra rows (dislocations) were introduced a few rows out from the centers to achieve more uniform spacing. The resulting pillow-like regions between lines, which would correspond to areoles in a cactus, are quadrilateral (ignoring the slight curvature) except in four locations midway between pairs of growth centers and offset in either direction from the line connecting the centers. In these four locations larger areas remain, bounded by six curved lines. An implementation in a thick paddle-like clay form is shown in Figure 11(b), where grooves were formed using both custom hand-made clay and wood tools.

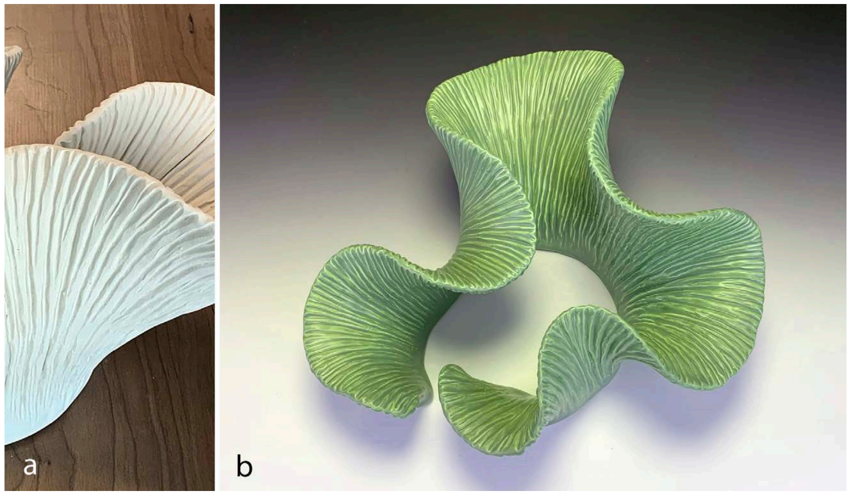

Figure 9: *Ceramic sculpture inspired by crested cactuses: a) detail of grooves made in a hyperbolic clay surface ; b) finished piece, "Groovy Hyperbolic Form" (2021).*

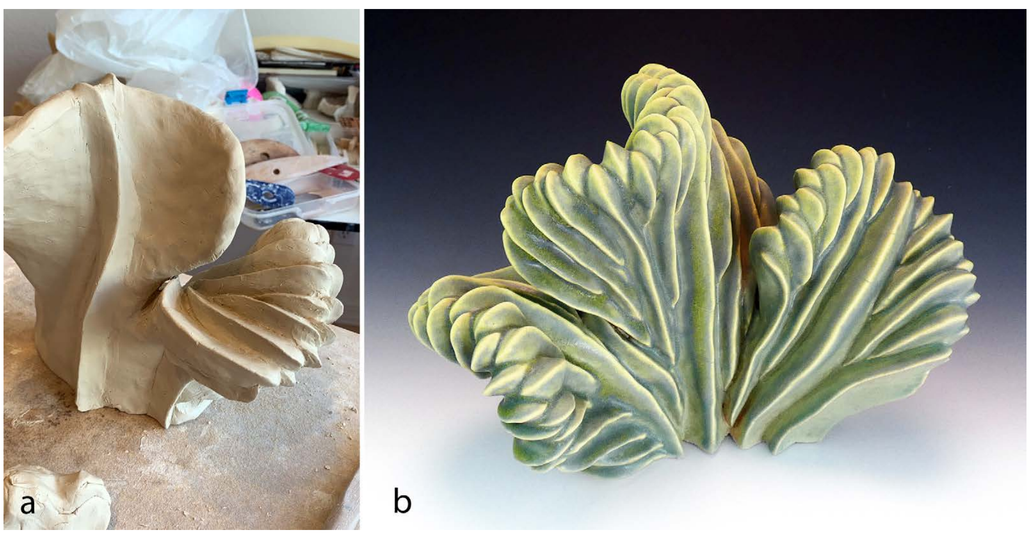

Figure 10: *Ceramic sculpture inspired by crested blue myrtle cactuses: a) construction method, with triangular strips attached to paddle-like forms; b) finished piece, "Crested Cactus Form" (2021).*

The form that cresting takes in globular cactuses (e.g., Figure 6) is distinct from that in columnar species and similar in surface structure to brain coral. The closely-packed surface features, which obscure the underlying structure, are related to space-filling curves. A clay sculpture inspired by these forms, with superlinear growth and a roughly hemispherical envelope, is shown in Figure 12. The curves were not planned out in advance, but were created as the piece was built by pinching a thick ring-like mass of clay, consciously avoiding a regular structure. I've previously demonstrated similar sculptures with carefully-planned-out fractal curve structure [10, 11].

Creating a sculpture of the sort shown in Figure 12 relies on being able to pinch and mold the surface to build in superlinear growth and doesn't allow the sort of dense packing observed in Figure 6. As a result, the features are narrow, creating a very different look from that in crested globular cactuses. Note also that the ridges of Figure 12 are all one continuous line. While a single unbroken growth front seems likely to form in a crested globular cactus, what meets the eye in Figure 6 is a collection of broken lines. The breaks are due to the growth front being folded too deeply in the overall envelope of the plant to be visible.

In order to better capture that look, I built another sculpture by applying thick strips of clay to the surface of a roughly hemisphere base (Figure 13(a)). The internal structure is obviously not similar to a crested cactus or brain coral, but the visible part reproduces it more faithfully. While the final piece, as shown in Figure 13(b), is somewhat similar in overall form to a crested globular cactus, the lack of ridges and spines keep it from evoking a cactus. Instead, smooth tubular biological entities such as worms or intestines come to mind.

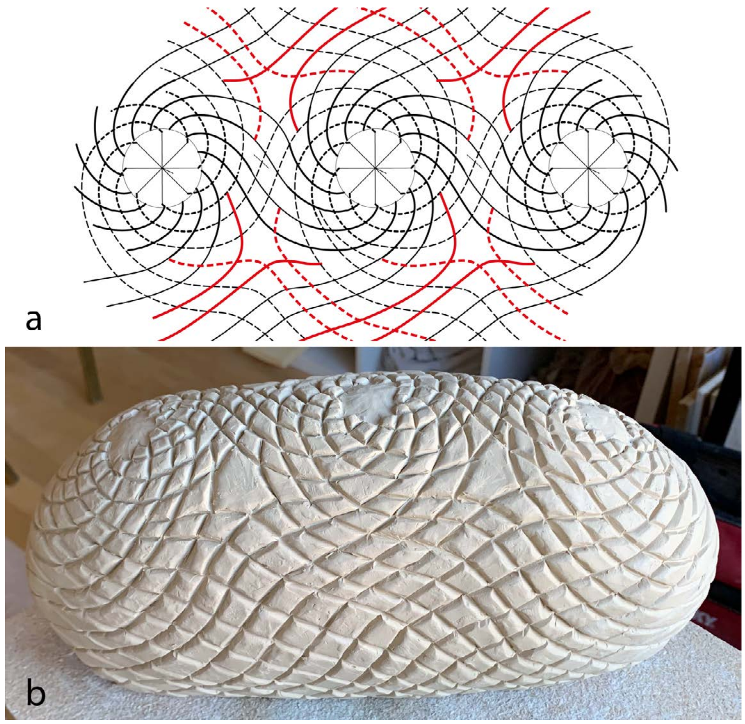

Figure 11: *A line of three interacting sets of phyllotactic spirals: a) design on paper, and b) a clay sculpture (2022, unfired).*

Figure 12: *Clay sculpture inspired by crested globular cactuses (2022, unfired): a) construction method, based on pinching a thick disk of clay; b) fully-built piece.*

Figure 13: *Clay sculpture inspired by crested globular cactuses (2022, unfired): a) construction method, with clay bands affixes to a hemispherical base; b) fully-built piece.*

Summary and Conclusions

I have described mathematical features evidenced in cactus structures and taken a closer look at plants exhibiting the growth abnormality known as cresting. Phyllotactic spirals, fractal structure, and hyperbolic character all play a role in these plants. Five clay sculptures inspired by crested cactus forms have also been shown. This work could be extended by sculpting more complex forms that better capture the drama exhibited by the cactus in Figure 7a or the intricacy of the cactus in Figure 6. A more detailed mathematical model of growth features resulting from a line of point growth sources could be developed as well. All of the photographs in this paper were taken by me, and all of plants are in my yard in Arizona unless otherwise noted.

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