Koos' Star Exists: Proof Outline

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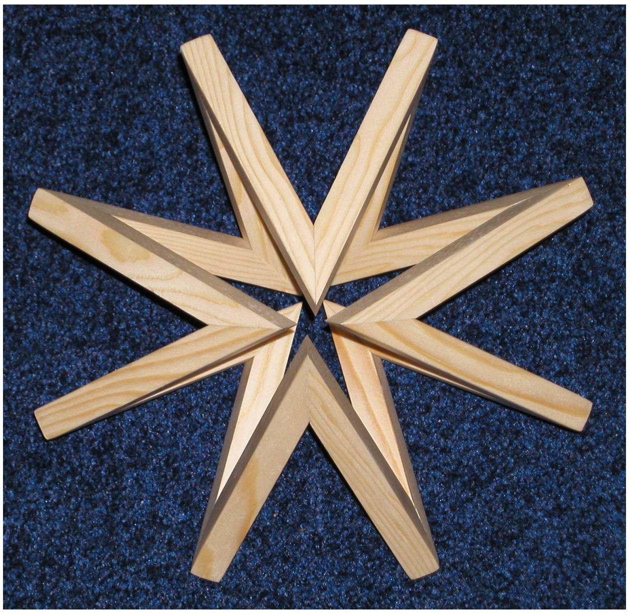

Figure 1: Koos' Star

The (narrow) claim (see Fig.1) is as follows.

> There exists a mathematical object with the following properties:
>
> 1. It is a closed polygonal path in 3D space of 16 segments.
> 2. All the segments have the same length.
> 3. All the angles α between adjacent segments are the same, and $0 < \alpha < \pi/2$.
> 4. All torsion angles ψ (see below) equal $\pi/2$ in absolute value.
> 5. The signs of the torsion angles are $(++--)^4$.

To define the torsion angle of a segment in a closed polygonal path, consider that segment BC and the two adjacent segments AB and CD. Then the torsion angle of BC is the directed angle from the plane spanned by ABC to the plane spanned by BCD, using the right-hand rule with respect to the directed rotation axis AB.

Here is the outline of a proof, with a bit more detail than in [1].

1. Relax the problem, by considering the 3D turtle program $KS(\phi)$:

   $$\Big(M(1); R(\frac{\pi}{2}); T(\phi); M(1); R(\frac{\pi}{2}); T(\phi); M(1); R(-\frac{\pi}{2}); T(\phi); M(1); R(-\frac{\pi}{2}); T(\phi)\Big)^4$$

   involving the following turtle commands operating on state (v, h, n) of position, heading, and normal vectors.

   - $M(d)$ move along h by distance d: $v \mapsto v + dh$;
   - $R(\psi)$ roll about h by angle ψ: $n \mapsto n\cos\psi - (n \times h)\sin\psi$;
   - $T(\phi)$ turn about n by angle ϕ: $h \mapsto h\cos\phi + (n \times h)\sin\phi$.

2. The goal now is to prove that a turn angle ϕ exists such that $KS(\phi)$ *closes properly*, that is, the turtle returns to its initial state after executing the program. Then $\pi - \phi = \alpha$ above, and the torsion angles equal $\pm\pi/2$ according to the pattern $(++--)^4$.

3. When a turtle program is properly closed, the shape of the path generated will not change when cyclicly rotating the commands. See Cyclic Permutation Congruence (CPC) Theorem in [2].

4. There are various algebraic properties of turtle commands (see [2]):

   $$\begin{array}{rcl} T(\phi_1 + \phi_2) & \equiv & T(\phi_1); T(\phi_2) \\ M(d); R(\psi) & \equiv & R(\psi); M(d) \end{array}$$

5. Rewrite $KS(\phi)$ as

   $$\begin{array}{ll} & \big(M(1); R(\frac{\pi}{2}); T(\phi); M(1); R(\frac{\pi}{2}); T(\phi); \\ & \ M(1); R(-\frac{\pi}{2}); T(\phi); M(1); R(-\frac{\pi}{2}); T(\phi)\big)^4 \\ \stackrel{c}{\equiv} & \qquad \{\ T(\phi) = T(\dfrac{\phi}{2}); T(\dfrac{\phi}{2}) \text{ and CPC Theorem }\} \\ & \Big(T(\frac{\phi}{2}); M(1); R(\frac{\pi}{2}); T(\phi); M(1); R(\frac{\pi}{2}); T(\frac{\phi}{2}); \\ & \ T(\frac{\phi}{2}); M(1); R(-\frac{\pi}{2}); T(\phi); M(1); R(-\frac{\pi}{2}); T(\frac{\phi}{2})\Big)^4 \\ \equiv & \qquad \{\ M(1); R(\psi) = R(\psi); M(1)\ \} \\ & \Big(T(\frac{\phi}{2}); M(1); R(\frac{\pi}{2}); T(\phi); M(1); R(\frac{\pi}{2}); T(\frac{\phi}{2}); \\ & \ T(\frac{\phi}{2}); R(-\frac{\pi}{2}); M(1); T(\phi); R(-\frac{\pi}{2}); M(1); T(\frac{\phi}{2})\Big)^4 \end{array}$$

6. Define the reflection $\mathit{refl}'(p)$ ($= \mathit{refl}$ in [1]) of a turtle program p by

   $$\begin{array}{rcl} \mathit{refl}'(M(d)) & = & M(d) \\ \mathit{refl}'(R(\psi)) & = & R(-\psi) \\ \mathit{refl}'(T(\phi)) & = & T(\phi) \\ \mathit{refl}'(p; q) & = & \mathit{refl}'(q); \mathit{refl}'(p) \end{array}$$

7. Reflection Lemma: The program $p; \mathit{refl}'(p)$ produces a path that is mirror symmetric with as reflection plane the plane passing through the final position after p, and perpendicular to the final heading after p. Moreover, if the turtle's attitude after $p; \mathit{refl}'(p)$ is h', n' (heading, normal), then $-h', n'$ is the mirror image of the initial attitude. (See [2, §5.2] for a similar lemma, and Appendix A below for details.)

8. Observe that for turtle program p, we have $\mathit{refl}'(\mathit{refl}'(p)) = p$.

9. We can write $KS(\phi) = (P(\phi); \mathit{refl}'(P(\phi)))^4$ with

$$P(\phi) = T(\frac{\phi}{2}); M(1); R(\frac{\pi}{2}); T(\phi); M(1); R(\frac{\pi}{2}); T(\frac{\phi}{2})$$

From now on, we leave the parameter of P implicit.

10. Now apply the Reflection Lemma twice:

    - to $P; \mathit{refl}'(P)$
    - to $\mathit{refl}'(P); P = \mathit{refl}'(P); \mathit{refl}'(\mathit{refl}'(P))$

11. Thus we find that KS starts as P, followed by a mirror image $\mathit{refl}'(P)$, which is in turn followed by its mirror image $\mathit{refl}'(\mathit{refl}'(P)) = P$.

12. When $KS(\phi)$ is properly closed, we see that it has those two reflections as symmetry operations. Hence, it has a rotational symmetry with a rotation angle that is twice the angle between the reflection planes.

    And conversely, when the angle between those two reflection planes equals $\pi/4$, program $KS(\phi) = \big(P; \mathit{refl}'(P)\big)^4$ will close properly.

13. Let $\theta_{KS}(\phi)$ be the rotation angle between the initial and final state (and in particular, its headings) of $P; \mathit{refl}'(P)$, as function of ϕ.

    Observe that (also see Fig. 2):

    - Function $\theta_{KS}(\phi)$ is continuous (also see Step 14).
    - $\theta_{KS}(0) = \theta_{KS}(\pi) = 0$
    - $\theta_{KS}(\pi/2) = 2\pi/3$ (this path walks in the simple cubic lattice)

    Hence, by the Intermediate Value Theorem, there exists a ϕ with $0 < \phi < \pi/2$ and also a ϕ with $\pi/2 < \phi < \pi$ such that $\theta_{KS}(0) = \pi/2$.

14. In fact, from P we can calculate (using Mathematica):

$$\begin{aligned} \cos 2\theta_{KS}(\phi) &= (3 + \cos 2\phi)/4 \\ \theta_{KS}(\phi) &= 2 \arccos((3 + \cos 2\phi)/4) \end{aligned}$$

Thus, we have the following two approximate solutions for ϕ: 49.9396° and 130.0604°. The latter corresponds to Koos' Star. It is now also clear that these two solutions sum to 180°.

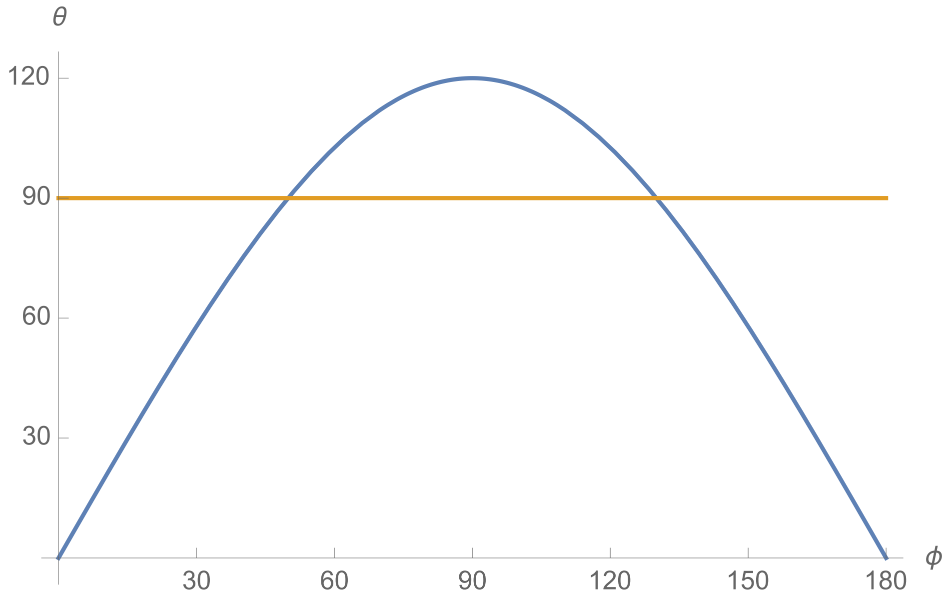

Figure 2: Plot (in blue) of rotation angle $\theta_{KS}(\phi)$ for $P(\phi)$ of Koos' Star

The equation to solve is

$$(3 + \cos 2\phi)/4 = \cos \pi/4 = \frac{1}{2}\sqrt{2}$$

It has as solutions

$$\phi = \pm\frac{1}{2}\arccos(-3 + 2\sqrt{2})$$

Summary (in hindsight)

One way of comparing our original approach in [3] to that of [1] is as follows. In 2009, we were looking for solutions to this equation in ϕ: *final state* of

$$\left(\left(M(1); R(\frac{\pi}{2}); T(\phi)\right)^2; \left(M(1); R(-\frac{\pi}{2}); T(\phi)\right)^2\right)^4$$

equals *initial state*, yielding a system of three complicated vector equations.

In 2021, we are looking for solutions to the equation in ϕ: *angle* $\theta/2$ between *initial heading* and *final heading* after

$$\left(T(\frac{\phi}{2}); M(1); R(\frac{\pi}{2}); T(\frac{\phi}{2})\right)^2$$

equals 45° (these headings are the normal vectors of the reflection planes). The latter is a single simple scalar equation, when using the vector dot product of initial and final heading to get $\cos\frac{\theta}{2}$. Also see Fig. 3.

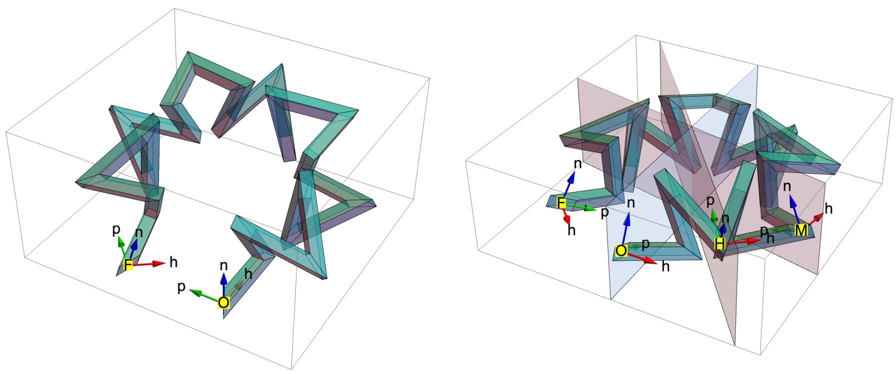

Figure 3: $(+++---)^4$: The 2009 view (left) and the 2021 view (right)

Generalization

This argument can be generalized. Let P equal

$$T(\frac{\phi_k}{2}); M(d_1); R(\psi_1); T(\phi_1); \ldots; M(d_k); R(\psi_k); T(\frac{\phi_k}{2})$$

for distances d_i and angles ψ_i, ϕ_i with $1 \leq i \leq k$. Then P defines a rotation angle θ, and $\big(P; \mathit{refl}'(P)\big)^n$ is properly closed if and only if $n\theta$ is a multiple of 2π. Moreover, this θ is a continuous function of each of the parameters.

References

[1] M. van Veenendaal and T. Verhoeff. "Pretty 3D Polygons: Exploration and Proofs." *Proceedings of Bridges 2021: Mathematics, Art, Music, Architecture, Education, Culture.* Phoenix, Arizona: Tessellations Publishing, 2021, p. (to appear)

[2] T. Verhoeff. "3D Turtle Geometry: Artwork, Theory, Program Equivalence and Symmetry." *Int. J. of Arts and Technology*, vol. 3, no. 2/3, 2010, pp. 288–319

[3] T. Verhoeff and K. Verhoeff. "Regular 3D Polygonal Circuits of Constant Torsion." *Bridges Conference Proceedings*, Banff, Canada, Jul. 26–30, 2009, pp. 223–230. http://archive.bridgesmathart.org/2009/bridges2009-223.html

A Reflection Lemma

We prove this by reduction to the reflection property of [2, §5.2]. In [2], the following definitions and properties are given.

- Turtle program transformer *rev* is defined by

$$\begin{aligned} rev(M(d)) &= M(-d) \\ rev(R(\psi)) &= R(-\psi) \\ rev(T(\phi)) &= T(-\phi) \\ rev(p;q) &= rev(q); rev(p) \end{aligned}$$

  It is an involution (its own inverse).

- Turtle program transformer *refl* is defined by

$$\begin{aligned} \mathit{refl}(M(d)) &= M(d) \\ \mathit{refl}(R(\psi)) &= R(-\psi) \\ \mathit{refl}(T(\phi)) &= T(\phi) \\ \mathit{refl}(p;q) &= \mathit{refl}(p); \mathit{refl}(q) \quad [\neq \mathit{refl}'(p;q)] \end{aligned}$$

  It is an involution.

- *rev* and *refl* commute: $rev \circ \mathit{refl} = \mathit{refl} \circ rev$.

- Half loop $\mathcal{H} = T(\pi); R(\pi) = R(\pi); T(\pi)$, with properties

$$\begin{aligned} \mathcal{H};\mathcal{H} &= \mathcal{I} \\ \mathcal{H}; M(d) &= M(-d); \mathcal{H} \\ \mathcal{H}; T(\phi) &= T(-\phi); \mathcal{H} \\ \mathcal{H}; R(\psi) &= R(-\psi); \mathcal{H} \end{aligned}$$

Let's define turtle program transformer *flip* by $\mathit{flip}(p) = \mathcal{H}; p; \mathcal{H}$. This flips the signs of all arguments in the commands in the program. This transformer commutes with both *rev* and with *refl*.

The turtle program transformer refl' defined under 6, satisfies

$$\mathit{refl}' = \mathit{refl} \circ \mathit{flip} \circ rev$$

The reflection property in [2, §5.2] states that

> the programs p and $\mathit{refl}(p)$ produce paths that are each other's mirror image w.r.t. the (x, y)-plane. This lemma should have been extended with the claim that the state after p, say (h, n), and the state after $\mathit{refl}(p)$, say (h', n'), are related as follows: h and h' are each other's reflection in the (x, y)-plane, and n and $-n'$ are each other's reflection in the (x, y)-plane.

Consequently (by property of $\mathcal{H}$, which equals the composition of two reflections: in (x, y)-plane and in (y, z)-plane),

> the programs p and $\mathcal{H}; \mathit{refl}(p); \mathcal{H}$ produce paths that are each other's mirror image w.r.t. the (y, z)-plane, and the final states are related as follows: h and $-h'$ are each other's reflection in the (y, z)-plane, and n and n' are each other's reflection in the (y, z)-plane.

Hence, by taking $\mathit{rev}(p)$ for p in the preceding, we have

> the programs $\mathit{rev}(p)$ and $\mathcal{H}; \mathit{refl}(\mathit{rev}(p)); \mathcal{H}$ produce paths that are each other's mirror image w.r.t. the (y, z)-plane, and the final states are related as follows: h and $-h'$ are each other's reflection in the (y, z)-plane, and n and n' are each other's reflection in the (y, z)-plane.

Now observe that $\mathcal{H}; \mathit{refl}(\mathit{rev}(p)); \mathcal{H} = \mathit{refl}'(p)$. So, we conclude

> the programs $\mathit{rev}(p)$ and $\mathit{refl}'(p)$ produce paths that are each other's mirror image w.r.t. the (y, z)-plane, and the final states are related as follows: h and $-h'$ are each other's reflection in the (y, z)-plane, and n and n' are each other's reflection in the (y, z)-plane.

And this is equivalent to the Reflection Lemma under 7.

B Mathematica Code

Here, we present the Mathematica code to obtain closed formulas. The turtle state is encoded in a list of three vectors: position, heading, and normal.

```
istate = {{0, 0, 0}, {1, 0, 0}, {0, 0, 1}}; (* initial state *)
{pos, hdg, nrm} = {1, 2, 3}; (* indices to extract parts of a state *)

move[d_][{position_, heading_, normal_}] :=
    {position + d heading, heading, normal}

turn[phi_][{position_, heading_, normal_}] :=
    {position, Cos[phi] heading + Sin[phi] Cross[normal, heading], normal}

roll[psi_][{position_, heading_, normal_}] :=
    {position, heading, Cos[psi] normal - Sin[psi] Cross[normal, heading]}

compose[{}][state_] := state
compose[{step_, t___}][state_] := compose[{t}][step[state]]
```

```
segment[d_, psi_, phi_] :=
    compose[{move[d], roll[psi], turn[phi]}]

factor[c_] := Switch[c, "+", +1, "-", -1] (* s is a string of signs *)

rollsignstoprogram[s_, d_, phi_, psi_] :=
    compose[Table[segment[d, factor[c] psi, phi], {c, Characters[s]}]]
```

A formula for the final position after $(+-)^3$ as function of turn angle *phi* is obtained by:

```
rollsignstoprogram["+-+-+-", 1, phi, Pi/2][istate][[pos]] // TrigReduce
```

which produces

```
{1/16 (16 + 26 Cos[phi] + 24 Cos[2 phi] + 21 Cos[3 phi] +  8 Cos[4 phi] + Cos[5 phi]),
 1/8 (6 Sin[phi] + 10 Sin[2 phi] + 6 Sin[3 phi] + Sin[4 phi]),
 1/16 (18 Sin[phi] + 16 Sin[2 phi] + 19 Sin[3 phi] + 8 Sin[4 phi] +  Sin[5 phi])
}
```

To find the formula for proper closure of $(++--)^4$, we define (cf. Step 9 above)

```
P[phi_] := compose[{turn[phi/2],
                    move[1], roll[Pi/2], turn[phi],
                    move[1], roll[Pi/2], turn[phi/2]}]

thetaKS[phi_] := 2 ArcCos[Dot[istate[[hdg]], P[phi][istate][[hdg]]] // TrigReduce]

thetaKS[phi] (* angle spanned by ++-- as function of phi *)
```

which produces

```
2 ArcCos[1/4 (3 + Cos[2 phi])]
```

Thus, we need to solve for *phi* to make this equal to $\pi/2$:

```
Solve[thetaKS[phi] == Pi/2, phi, Reals]
```

which produces

$$\left\{\left\{\text{phi} \to \boxed{\tfrac{1}{2}\left(2\pi c_1 - \cos^{-1}\left(2\sqrt{2}-3\right)\right) \text{ if } c_1 \in \mathbb{Z}}\right\}, \left\{\text{phi} \to \boxed{\tfrac{1}{2}\left(\cos^{-1}\left(2\sqrt{2}-3\right) + 2\pi c_1\right) \text{ if } c_1 \in \mathbb{Z}}\right\}\right\}$$

Numerically:

```
With[{phi = 1/2 ArcCos[-3 + 2 Sqrt[2]]/Degree}, {phi, 180 - phi}] // N
```

yielding

```
{49.9396, 130.06}
```