Constructivist Art based on the Mandelbrot Set

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Abstract

Abstract: the artworks explore the geometric meaning of the basic computation step whose iteration is known to produce the Mandelbrot set. The work comprises web-apps and canvases with wooden panel constructions.

Introduction

Programs of the Mandelbrot set produce beautiful images based on a computation step $z \rightarrow z^2 + c$. The formula of this step is simple enough, yet there is a gap between seeing the formula and grasping it by intuition. That is because formula works in the complex number system, not with real numbers.

Talking about numbers, I just turned 66 and am now an emeritus professor, so I launched my own modest “lab” to explore art, fashion, math, and creative programming. As a birthday present, I was given the beautiful book by Peitgen and Richter [6], which showed me the richness of the Mandelbrot set again. I got under its spell. In the preface, I read Galileo’s words, that the grand book (the Universe) is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures. The geometry of the Mandelbrot computation step became my first lab project, Figure 1 gives a first impression.

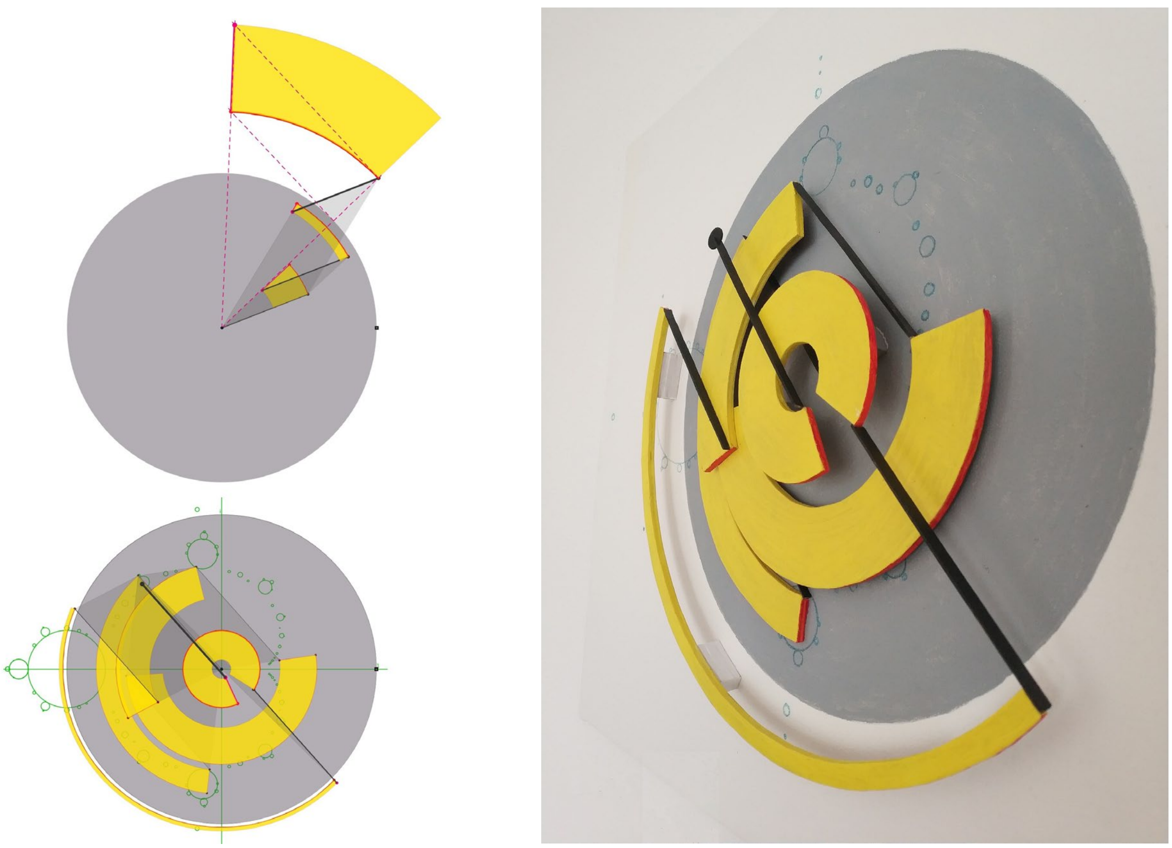

Figure 1: *Iterations for different initial values of c and implemented Artwork (wood, 50x50cm).*

In the early days when the public was informed about Mandelbrot's new paradigm, this gap between the formula and the intuition got some attention, for example in the animation films from 2008 by Leys, Ghys, and Alvarez presented by Adrian Douady [2]. Today, computers and displays allow for deeper zooms and higher resolutions, but that does not help to fill the gap. The artworks in this paper result from my endeavor to turn the basic computation step into a constructive geometric step, re-addressing the gap. The artwork in this project comes in several forms. First are the interactive programs (openprocessing.org/sketch/1063319 and openprocessing.org/sketch/1057072) to play with the computation steps and see their geometric nature in action. The second form are panels with jig-sawed wooden pieces showing the path of a point *z*. Based on Figure 2 (left and right), I made canvas and wood artworks of size 50x50cm.

Towards an Intuition of Squaring

The innocent looking formula from the first line of this article is not as easy as it appears, because *z* and *c* are complex numbers. They obey all rules of real-number arithmetic, but there is a little stowaway i. It is inaccurately named "imaginary". Its rule $i^2 = -1$, invented for reasons of algebra, challenges intuition. My program, producing the images of Figure 2, works in Processing, having no built-in i. So I coded complex numbers as coordinate pairs (x,y), and i as (0,1). First, look at squaring (the "+ *c*" comes later). Squaring is a special case of multiplication, the rule for which I could copy from the algebra books to my code[1].

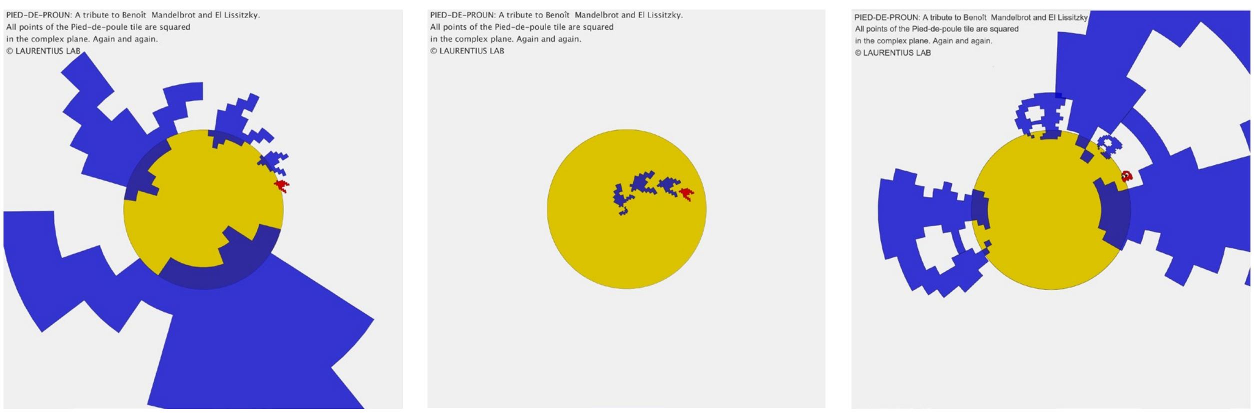

Figure 2: PIED-DE-PROUN, *successive squarings of a tile. Left: a Pied-de-poule tile (fashion pattern [7]). Center: the same tile inside the unit circle. Right: tile inspired by Toru Iwatani's Pacman game.*

Playing with the interactive program, I could strengthen my intuition of squaring. It is fun to see the tiles chasing, one after the other, while increasing their distance. The chain of tiles can turn inward, outward, or explode over the unit circle (try on openprocessing.org/sketch/1063319). Squaring a complex number amounts to doubling its angle and squaring its modulus (the distance to the origin). I knew this and I looked up the proof again[2], but the program gives me another, more experiential, understanding. For the graphic design, I deployed the basic tile of the well-known Pied-de-poule pattern, known as a source of fashion and works of art [7]. The result made me think of the 1920s Suprematist and Constructivist art, which is well-

[1] Multiplication of complex numbers, coded as vectors (a, b) and (c, d), goes by the rule $(a, b) \times (c, d) = (ac - bd, ad + bc)$. Squaring is $(x, y)^2 = (x^2 - y^2, 2xy)$. In my program, *x* and *y* are floating point numbers.

[2] My old intuition was about AC circuits. For cascaded amps, one can multiply the amplification factors, add-up the phase shifts. Mathematically, if $z = (x, y)$ then $\arg(z)$ denotes the angle from the positive *x*-axis to the vector *z* and its modulus $|z|$ is $\sqrt{x^2 + y^2}$. The theorem $\arg\big((a, b) \times (c, d)\big) = \arg(a, b) + \arg(c, d)$ and $|(a, b) \times (c, d)| = |(a, b)| \times |(c, d)|$ follows by trigonometry [1]. So $\arg(x, y)^2 = 2\arg\,(x, y)$, $|(x, y)^2| = (|x, y|)^2$. Sometimes the geometry of multiplication is explained via $r_1 e^{i\vartheta} \times r_2 e^{i\vartheta_2} = r_1 r_2\, e^{i(\vartheta_1 + \vartheta_2)}$ but that shifts the problem to obtaining an intuition for complex exponentiation $e^{i\vartheta} = \sum_{n=0}^{\infty} (i\vartheta)^n / n!$ (Howie [3], p.24).

represented in the Van Abbe museum Eindhoven. The museum was forbidden terrain because of a pandemic, but I could sneak-in "virtually" at **vanabbemuseum.nl** and see the work of El Lissitzky (1890–1941), who called his works "proun" (sounds like pro-oon) [5].

In my PIED-DE-PROUN, as I call my program now, each of the red tile's 34 corner points (and some interpolating points) are squared in complex number arithmetic. The resulting points define another tile, colored blue. The blue tile is squared again, and again. Squaring has the effect of approximately doubling the red tile's radial size (when r is close to 1, we know $r^2 \approx 1 + 2r$). The tile is also stretched in the tangential direction (each φ is mapped to 2φ). Any small figure near the circumference of the unit circle is stretched in both directions, so its surface area is about the square of the original (aha, here it squares indeed).

Geometric Squaring and Addition

The Mandelbrot set is known for its complexity, which arises from the interplay of the sub-steps of $z \rightarrow z^2 + c$, which are threefold. Two sub-steps describe squaring a complex number, viz. $\varphi \rightarrow 2\varphi$ and $r \rightarrow r^2$, the third sub-step is adding c. In traditional Mandelbrot exploration tools, such as "Iterates for the Mandelbrot Set" (demonstrations.wolfram.com) by Felipe Dimer de Oliveira, the sub-steps are hidden, as is the algebra under the hood. By contrast, I want to visualize the sub-steps and I want to do it with geometry, not algebra.

The geometric construction for addition on the complex plane works like adding vectors. The line from the origin-point to the point c, is moved in a parallel manner to the end of the vector to which it is to be added. The characteristic parallelogram is well-known. Is there a similar construction for squaring complex numbers? A geometric representation of multiplication can be found in older textbooks such as by N.G. De Bruijn [1] p.242 (Figure 3, the triangles O1α and Oβ(αβ) are similar).

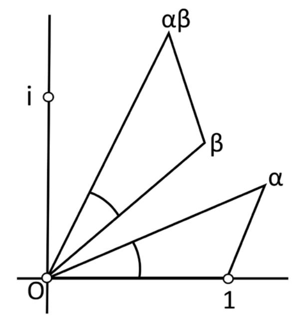

Figure 3: *Geometric interpretation of complex number multiplication.*

To display the sub-steps, I decided to use this wonderful tool Jun Hu and I already used in other (Bridges) projects: turtle graphics [4]. As the turtle moves forward, it traces the orbit of the point z, but moving in small hops: one for each sub-step. The first and second sub-steps implement complex squaring: *doubling the angle* and *squaring the distance* (to O). So if z is given by r and φ then the turtle moves to the point having angle 2φ and distance to O being r^2. The sub-steps appear as bold red lines in Figure 4 (left and center). Alternatively, the two sub-steps could be done the other way around, so there is a (curved) commuting diagram, which I think should stand out. It is colored yellow in Figure 4 (center).

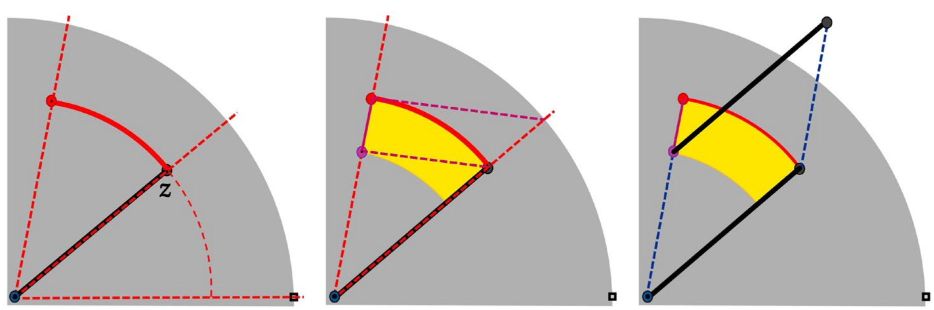

Figure 4: *The 1st iteration step, decomposed in 3 sub-steps. From left to right: doubling the angle, constructing r^2, and adding c. The grey quadrant is part of the unit circle.*

Thus I found the geometric construction for squaring. The characteristic figure is not a traditional parallelogram but this four-sided arc: two concentric pieces of circles and two radial lines. It is similar to the parallelogram, in the sense that one can follow either of two paths: *doubling the angle* and *squaring the* *distance*. The dashed red lines are auxiliaries, forming the similar triangles such that $r^2 : r = r : 1$, inspired by the diagram in De Bruijn's book [1]. The two similar triangles are now homothetic. Even the step $r \rightarrow r^2$ is no longer a matter of (real) number arithmetic, but is made by geometry.

The program PROUN2021

The next program, PROUN2021 implements these sub-steps and allows the user to iterate them. The program presents an interactive experience (like the previous, PIED-DE-PROUN). The user can choose the constant *c* by cursor position, and proceed to develop the iterations and/or play with different *c*. After every mouse-movement, the constructions of Figure 4 will show-up but disappear after a few seconds in order to not clutter-up the big picture. The user can also toggle to see the main bulbs (green in Figure 1). The green parts indicate the bulbs of the Mandelbrot set: if the iteration starts in such an area, it will eventually return to the same green area. The path of the turtle clarifies what is going on: the point *z* tends to rotate around the origin. An ever accelerating counterclockwise wind is blowing. At the same time, the point is subject to a centripetal force towards the origin when inside the unit circle (centrifugal once outside). The nature of the Mandelbrot set becomes apparent: occasionally, the addition step un-does the centripetal/centrifugal force. That is why the path of the successive points may converge, run in orbits or finally escape to infinity. The end of the path is very sensitive to the initial condition (adding iterations makes it more sensitive).

PROUN2021 is written in Processing using turtle graphics. Openprocessing.org/sketch/1057072 is a p5.js version (no download, enable Javascript), generating an unlimited number of designs, one for each combination of *c* and the chosen number of iterations.

I implemented one generated design on a canvas, using wooden panels and jig-sawed arcs, painted, and positioned at different heights (each yellow arc is positioned on top of those from the earlier construction steps). The size of the physical artwork is 50x50cm, see Figure 1 (right). It shows the path of *z* when $c = -0.5465+0.6126i$. This *c* is in the period doubling secondary bulb attached to the bulb known as 2/5. An observer can follow the fivefold iteration by painted red lines (the sides of the yellow panels) and black lines (thin wooden sticks, all parallel and the same length, *c*).

Conclusions

Everything needed for iterating $z \rightarrow z^2 + c$ was already known to Euclid, but it was not until Van der Pol and Armstrong heard noise coming from their electronic oscillators that chaos in the modern understand-ding was perceived. Using digital computers, Mandelbrot opened our eyes and now I arrived in this play-ground, my lab to explore the nature of chaos in a mix of old and new technologies.

References

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